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Topologies on the Space of Lorentz Metrics and Stability of Global Properties of a Space-time Manifold

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Abstract
Following the work of researchers spread over more than four decades, we describe a number of topologies on the space of Lorentz metrics on a space-time manifold and using some of these topologies, especially Whitney C^r - topology, we discuss stable and unstable global properties of space-times. We also discuss a new topology introduced by Noldus on this space.

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1. Introduction

Each Lorentzian metric defined on a four dimensional manifold satisfying Einstein field equations represents a space-time. It determines the geometry (cone structure) and causal structure of space-time. Until early 1960's, researchers working in the General Theory of Relativity used to study only special solutions of Einstein field equations. However, with the development of causal structure theory by Hawking and Penrose, and later by other co-workers (cf. Hawking-Ellis [1], Joshi[2]), a new avenue was opened in the study of space-time structure, which was global in nature. Methods were used from topology and global differential geometry to study global structure of space-time which ultimately led to the proof of singularity theorems [1,2]. Hierarchy of a number of causality conditions was discovered in these studies which were mainly global properties of space-time manifold. (for recent review of these results, see Minguzzi [3] and Chrusciel [4]). The important question that then arose was about establishing stability of these properties in a suitable topological sense. For this it became necessary to define an appropriate topology on the space of Lorentz metrics on a given space-time manifold.

Thus, Hawking [5], Lerner [6], Beem and Ehrlich [7], Beem [8] etc. defined a Whitney C^r - topology and other topologies on this space and studied stability of global properties of space-time.

As we shall see below, definition of stability of a property requires the concept of an open subset of the space of Lorentz metrics. This in turn depends on the topology chosen on the space of such metrics. In this article we first discuss different ways of defining topology on this space and then discuss global properties which are stable or unstable under some of these topologies. Finally we remark on some interesting possibilities and future work to be carried out along these lines.

2. Topologies on the space of Lorentz metrics

Let M be a C^∞ Hausdorff paracompact four dimensional manifold. We further assume that M is non-compact without boundary. Let Lor(M) denote the space of C^r Lorentz metrics on M and Con(M) denote the quotient space formed by identifying all pointwise globally conformal metrics $g_1 = \Omega g_2$ on M where $\Omega : M \to (0, \infty)$ is a smooth function. Let $\pi : Lor(M) \to Con(M)$ denote the natural projection map which takes each Lorentzian metric g on M to the set $\pi(g) = [g]$ of all Lorentzian metrics on M which are pointwise globally conformal to g. Given $[g] \in Con(M)$, set C (M,g) $= \pi^{(-1)}([g]) \subseteq Lor(M)$. In general relativity, we say that a curvature condition or a causality condition for a space-time (M, g) is C^r - stable in Lor(M) (respectively Con(M)) if the validity of the condition for (M, g) implies the validity of the condition for all g_1 in a C^r - open neighbourhood of g in Lor(M) (respectively Con(M)). More generally, a stable condition for a set of metrics is one which holds on an open subset of such metrics.

In order that the notion of *open set* is well-defined, we need a topology on Lor (M). This can be done in more than one way, as follows :

To define a topology on Lor(M), we need the concept of a *distance* between two Lorentz metrics on M. For this, following S. W.Hawking [5], we can consider a positive definite metric e on M (this can always be done, since M is paracompact). This metric can then be used to define covariant derivatives of tensor fields on M and also to measure the magnitude of such tensor fields and their derivatives. With this, we can define how near together the derivatives of two metrics are at each point of M. We have to consider different possibilities.

1. The metrics can be required to be near only on compact regions of the manifold and the behaviour near infinity is unrestricted. This means, if g is a Lorentz metric, U a compact set of M and $\epsilon_i (0 \leq i \leq r)$ a set of continuous positive functions

on M, the neighbourhood $B(U, \epsilon_i, g)$ of g can be defined as the set of all Lorentz metrics whose i^{th} derivatives $(0 \leq i \leq r)$ differ from those of g by less than ϵ_i on U. The set of all such $B(U, \epsilon_i, g)$ for all U, ϵ_i and g form a sub-basis for the C^r *compact-open topology* for Lor(M). In other words, the open sets in this topology are unions and finite intersections of the $B(U, \epsilon_i, g)$.

2. The requirement that the sets U should be compact can be removed and U can be taken to be M. This means that nearby metrics must be nearby everywhere and must have the same limiting behaviour at infinity. This topology is called *open topology* for Lor(M).

3. Define the set $F(U, \epsilon_i, g)$ as the set of all metrics whose i^{th} derivatives differ from those of g by less than ϵ_i and which coincide with g outside the compact set U. The neighbourhood $B(\epsilon_i, g)$ is then defined as the union of $F(U, \epsilon_i, g)$ for all compact sets U. The neighbourhoods $B(\epsilon_i, g)$ form a sub-basis for the fine topology on Lor(M). This topology is finer than the open topology, which in turn is finer than the compact open topology. This means that there are more open sets in the fine topology than in the open topology and still more than in the compact open topology.

Classical C^r derivatives can be replaced by weak / generalized derivatives and we can obtain Sobolev W^r-topologies by demanding that instead of requiring the difference between the derivatives to order r of two nearby metrics to be small (point-wise), we require the integrals of squares of the differences of weak derivatives to order r to be small. The squares and the integrals are here defined with respect to positive definite metric e on M. Thus a C^r-tensor field is also a W^r field, and by using Sobolev embedding theorem it follows that a W^s tensor field for $s > \frac{n}{2} + r = \frac{4}{2} + r = r + 2$ say $s = r + 3$, is a C^r- field. This means that a W^{r+3} topology is finer than the corresponding C^r- topology which in turn, is finer than the W^r - topology (bigger norm - topology has more number of open sets than smaller norm - topology). W^r- topologies play an important role in the Cauchy problem in General Relativity (cf Hawking and Ellis [1]).

The topologies discussed above do not use the specific properties which emerge from the symmetry and signature of a Lorentzian metric.

The topology respecting the causal structure is defined by Bombelli, Lee, Meyer and Sorkin [9] as follows:

They use the fact that causal structure and conformal structure are the same when the Lorentzian metrics are future and past- distinguishing. They define a set of functions which compare the volume elements and causal structure of two metrics g and $\tilde{g}$ separately. When restricted to C^2- future and past-distinguishing metrics, this topology becomes Hausdorff. This means that for every point p, there exists a unique maximal geodesic with starting point p and initial direction $X_p \in T_pM$.

This geodesic depends continuously on p and X_p i.e. $\forall\ (p, X_p) \in TM$ and for every open neighbourhood N of $exp\,(p, X_p)$ in M, there exists an open neighbourhood U in TM such that $exp\ (q, X_q) \in N, \forall\ (q, X_q) \in U$ whenever $exp\ (q, X_q)$ is defined. This definition also satisfies the condition that for any diffeomorphism $\psi, d(g, \tilde{g}) = d(\psi^* g, \psi^* \tilde{g})$. Same condition is satisfied by Noldus' definition of the metric on Lor(M).

The definition of topology given by Bombelli and Meyer [10] is as follows:

One can find pseudo-distances which distinguish between metrics with different volume elements at $p \in M$. Using such a distance between local volume elements, they construct a distance between metrics in a given conformal class by

$$d^M(g, \tilde{g}) = \sup_{x,y \in M} \mid log \frac{\sqrt{-g(x)}}{\sqrt{-\tilde{g}(x)}} \mid$$

Then, they define a family of non-local functions, characterizing the fact that causal structures of two arbitrary metrics g and $\tilde{g}$ agree down to the volume scale λ, by

$$d^C_\lambda(g, \tilde{g}) = \sup_{x,y \in M, V(A) > \lambda} \frac{V(A \Delta A')}{V(A \cup A')}$$

where A is the Alexandrov neighbourhood defined by x and y in the metric g and A' is the corresponding neighbourhood in the metric $\tilde{g}$. Here V (R) denotes the volume of a region $R \subset M$ in the metric g and $A \Delta A'$ denotes the symmetric difference $(A \sim A') \cup (A' \sim A)$. We note that for a fixed value of λ, d^C_λ is not symmetric, nor does it satisfy the triangle inequality. However, the set of functions $\{d^v; d^C_\lambda; \lambda \in (0, \infty)\}$ induces a uniform structure on Lor(M).

More recently, Noldus [11] has defined a *new topology* on Lor(M). Noldus has generalized the above definition of *metric* function $d^C_\lambda(g, \tilde{g})$ and discussed uniform structure in details, which we discuss below:

Here, the main issue is to construct a fully diffeomorphism - invariant metric topology. Noldus modifies the work of Bombelli et al and tries to solve this problem *as much as possible.* His method is functional analytic. Two important concepts needed are:

(i) the choice of topology of the diffeomorphism group (Schwartz topology) and

(ii) the concept of amenability.

We discuss below some aspects of Noldus work and state his results.

Consider the class $\mathcal{H}_g$ defined for a Lorentz metric g as in Szabados [12]:

$\mathcal{H}_g = \{F \cap P /$ F is a future set and P is a past set$\}$. Szabados proves that all sets in $\mathcal{H}_g$ are Lebesgue measurable. Further, if m is a four-dimensional Lebesgue measure, then for any set $H \in \mathcal{H}_g$, we have $m(H) = m(\widetilde{H}) = m(\dot{H})$. Here $\widetilde{H}$ denotes closure of H and $\dot{H}$ denotes interior of H. We also note that $\mathcal{H}_g$ is not closed with respect to the union and has the property that $(F \cap P)^c = F^c \cup P^c$ equals the union of two elements of $\mathcal{H}_g$.

Let Ł be the σ -algebra of all Lebesgue measurable sets. Noldus defines two topologies on Ł by giving a base. Let A be an open subset and C a closed subset of M satisfying the property that $\bar{\dot{C}} = C$. (This is not true in general). The open set $B(A, C)$ is defined as follows:

$S \in B(A, C)$ iff $\dot{C} \subset \dot{S} \subset A$. It can be verified that these sets form a basis for the topology $\mathcal{T}_1$ on Ł.

The second topology $\mathcal{T}_2$ is defined as follows:

The open sets $B(A, C)$ consist of the following elements:

$S \in B(A, C)$ iff $C \subset \dot{S} \subset \bar{S} \subset A$.

Noldus thus gives the following definition.

Definition : A Lorentz metric g is of class $\mathcal{A}$ if and only if the map $A : M \times M \to \mathcal{H}_g : (p, q) \mapsto A(p, q) \subset \mathcal{H}_g$ is measurable with respect to one of the Borel σ-algebras defined by topological spaces $(\mathcal{H}_g, \mathcal{T}_1), (\mathcal{H}_g, \mathcal{T}_2)$.

Here $A(p, q) = J^+(p) \cap J^-(q)$ is the Alexandrov set. This class $\mathcal{A}$ contains all globally hyperbolic metrics. Alexandrov sets are compact for globally hyperbolic metrics. However for class $\mathcal{A}$, Alexandrov sets are not necessarily closed.

Noldus modifies definitions of Bombelli, Lee, Meyer and Sorkin [9] as below:

Definition : Let $g, \tilde{g} \in Lor(M)$. For each $p, q \in M$, let $A(p, q)$ and $\tilde{A}(p, q)$ be the Alexandrov sets for g and $\tilde{g}$ respectively. Define

(i) $$\alpha_{g\tilde{g}}(p, q) = \begin{cases} \dfrac{M(A(p, q) \Delta \widetilde{A}(p, q))}{M(A(p, q) \cup \widetilde{A}(p, q))} & \text{if } 0 < M(A \cup \widetilde{A}) < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $W_i \subset W_{i+1} \subset M, \cup_{i \in N} W_i = M$ and the closure of W_i is compact. Let $F : R^+ \to R^+$ be an increasing function such that $\exists\, \sigma \geq 1$ satisfying $f(ax) \leq a^\sigma f(x)$, $\forall a \geq 1$ and $x \in R^+$. Define following set of functions,

$$d^i_{cau}(g, \tilde{g}) = \frac{1}{f(M(W_i))} \int_{W_i \times W_i} \alpha_{g,\tilde{g}}(p, q) dM(p) dM(q)).$$

(iii) $$d^i_{vol}(g, \tilde{g}) = \sup_{p \in W_i} |log(\frac{\sqrt{-|g(p)|}}{\sqrt{-|\tilde{g}(p)|}}|$$

So this is a pseudo-distance which measures the difference in volume elements.

(iv) $d^i_{geo}(g,\tilde{g}) = \sup_{p,q\in W_i} |\lambda(p,q) - \widetilde{\lambda}(p,q)|$ where $\lambda(p,q)$ is zero when q is not in $J^+(p)$ and otherwise it equals $\sup_{\gamma\in C(p,q)} L(\gamma)$. (See Hawking and Ellis [1] for definition of L. Hawking and Ellis also prove that when g is globally hyperbolic, λ is continuous in p and q. Moreover $\lambda(p,q)$ equals the length of a non-space like g-geodesic curve from p to q. Thus $d^i_{geo}(g,\tilde{g})$ is clearly a pseudo-distance).
Noldus has proved the following result.

Proposition : If g is future and past-distinguishing and $\tilde{g}$ is a strongly causal metric which is not conformally related in $p \in M$, then $\exists\ i_0 \in N$ such that $\forall\ j \geq i_0$, $d^j_{cau}(g,\tilde{g}) > 0$. Thus the function d^i_{cau} distinguishes between conformally inequivalent strongly causal metrics.

The pseudo-distance d^i_{vol} compares the volume forms defined by both Lorentz metrics. This is used by Noldus to symmetrize the distance functions. The last metric compares the geodesic length between the points p and q. This gives us some information about the shape of Alexandrov sets.

Noldus has also proved that if g and $\tilde{g}$ are globally hyperbolic then the function $\alpha_{g\tilde{g}}$ is almost everywhere continuous.

We now give some introduction about uniformities and topologies and discuss how Noldus relates his definition of distance between Lorentz metrics with uniformities, thus giving a new topology on Lor(M):

Let (X,d) be a topological space where d is a pseudo-distance and $\mathcal{T}$ be the corresponding locally compact topology. We know that the family of open balls $B_{1/n}(p)$ with radius $\frac{1}{n}, n \in N$, around p, is a countable basis for $\mathcal{T}$ at p. Let I, J denote index sets and let $C = \{A_i/A_i \in \mathcal{T}, i \in I\}$ and $D = \{B_j/B_j \in \mathcal{T}, j \in J\}$ be open covers of $(X,\mathcal{T})$. C is said to be finer than D or C is a refinement of D or $C < D$ if and only if $\forall i \in I$, $\exists\, j \in J$ such that $A_i \subsetneq B_j$.
We now define some operations on the family of covers $\mathbf{C}(X,\mathcal{T})$:
(i) Let C and D be as above . Define

$C \wedge D = \{A_i \cup B_j / A_i, B_j \in \mathcal{T}, i \in I, j \in J\}$.

$C \wedge D$ is a cover of $(X,\mathcal{T})$ and $(\mathbf{C}(X,\mathcal{T}), \wedge)$ is a commutative semi-group.

(ii) For $A \subset X$, the star of A with respect to C is defined as follows :

$$St(A,C) = \bigcup_{A_i\in C : A\cap A_i \neq \phi} A_i$$

(iii) C^* , the star of C, is then defined as

$C^* = \{St(A_i, C)/A_i \in C\}$

We note that $C < C^* < C^{**} \cdots$ and if I is finite, then $\exists\, n \in N$ such that after n, star operations C becomes the trivial cover.

Using the collection of open balls as basis for the topology, we define elementary covers $C_n,\ n \in N$ as follows.

$C_n = \{B_{1/n}(p)/p \in X\}$.

Using these, we can define a subset **U** of $\mathbf{C}(X, \mathcal{T})$.

$\mathbf{U} = \{C \in \mathbf{C}(X, \mathcal{T})/\exists\, C_n$ such that $C_n < C\}$.

Then the family **U** satisfies the following properties:

(i) If $C \in \mathbf{U}$ and $C < D$, then $D \in \mathbf{U}$

(ii) if $C, D \in \mathbf{U}$, then $C \wedge D \in \mathbf{U}$

(iii) if $C \in \mathbf{U}$, then $\exists\, D \in \mathbf{U}$ such that $D^* < C$.

These properties are taken as a definition of *Uniformity*:

Definition : Let X be a set. A cover C is defined as

$C = \{A_i/A_i \subset X, i \in I\}$ such that $\bigcup_{i \in I} A_i = X$.

A collection $\mathcal{U}$ of covers is called a *uniformity* for X if and only if

(i) $C \in \mathcal{U}$ and $C < D$, then $D \in \mathcal{U}$

(ii) $C, D \in \mathcal{U}$, then $C \wedge D \in \mathcal{U}$

(iii) $C \in \mathcal{U}$, then $\exists\, D \in \mathcal{U}$ such that $D^* < C$ where all definitions of $<, \wedge$ and $*$ are independent of $\mathcal{T}$.

It is then well-known that any uniformity can be generated by a family of pseudo-distances. This implies that a uniformity defines a topology. This topology is the *new topology* on Lor(M). Further details can be found in Noldus [11].

As remarked by Noldus, the topology on Lor(M)/Diff(M) constructed in [10] is unique, however there exist many pseudo-distances which might generate this uniform topology. It is not known whether the topology on Lor(M)/G as constructed by Noldus is uniquely determined or not. It depends on the generating pseudo-distance of the uniform topology on Lor(M). However, when M is compact with boundary, one can also take the quotient Lor(M)/Diff(M). However, Noldus topology has much better continuity properties with respect to group actions. Another thing is, one can also raise the question whether the topology on Lor(M) is locally arcwise connected or not. Diff(M) is locally arcwise connected in the $\mathcal{FD}$ topology (Refer [13] for definition) so we have that for ϕ sufficiently small and $g \in Lor(M)$ that $\phi^* g$ is arcwise connected to g by a path in Lor(M) which corresponds to a path in Diff(M) from the identity to ϕ.

Finally, though Noldus topology is a generalization of topology given in [10], there is no specific direct relationship between this topology and topologies defined earlier on Lor(M). It would be interesting to investigate such relationships. However, in some special cases, say for dimension 2, Noldus topology is the same as Euclidean topology.

Thus, so far, we have discussed the recent results regarding the topologies that can be defined on Lor(M). Which of these topologies should be used in a given situation, depends on the properties one wishes to consider.

In the next section, we consider stability and instability of some global properties of a space-time with respect to some of the topologies discussed above. We shall also see how stability of a certain property can change to instability if one considers a different topology.

3. Stable and unstable properties

Consider the following causal properties :

Stable causality : We say that the stable causality condition holds on a space-time M if the space-time metric (Lorentz metric) g has an open neighbourhood in the C^0 open topology such that there are no closed timelike curves in any metric belonging to the neighbourhood.

Using C^0 topology here will not make any difference, but we could not use compact open topology because in this topology each neighbourhood of any metric contains closed timelike curves. Thus, stable causality condition means that one can expand the light cones slightly at every point without introducing closed timelike curves.

Global hyperbolicity : A space-time M is said to be globally hyperbolic if it is strongly causal and for any two points p,q in M, $J^+(p) \cap J^-(q)$ is compact.

We refer to [1,2] for detailed discussion of this property and its equivalent versions. As mentioned above, a property defined on (M, g), $g \in Lor(M)$, is said to be C^r stable if it holds on a C^r open subset (of Lor(M)) containing g. Similarly, a property defined on (M, g), $g \in Lor(M)$, which is invariant under the conformal relation is said to be conformally stable if it holds for an open set of equivalence classes in the quotient (or interval) topology (cf. Geroch [14]) on Con(M). The continuity of the projection map π implies that any conformally stable property defined on Lor(M) is also C^0 stable on Lor(M). Moreover, since the fine C^r topology is strictly finer than the fine C^s topology on Lor(M) for $r > s$, any conformally stable property defined on Lor(M) is also C^r stable for all $r \geq 0$.

Thus we have the following results (cf. Hawking [5], Geroch [14], Lerner[6], Beem[8]; also see the book [15] by Beem, Ehlrich and Esley for detailed discussion and proofs) :

1. Stable causality is conformally stable and hence also C^r stable in Lor(M) for all $r \geq 0$.
2. Global hyperbolicity is conformally stable and hence also C^r stable in Lor(M)

for all $r \geq 0$.
3. If (M,g) is a Lorentzian manifold such that $g(v,v) \leq 0$ and $v \neq 0$ in TM imply $Ric(g)(v,v) > 0$, then there is a fine C^2 neighbourhood U(g) of g in Lor(M) such that for all $g_1 \in U(g)$, the relations $g_1(v,v) \leq 0$ and $v \neq 0$ in TM imply $Ric(g_1)(v,v) > 0$. This shows that energy condition is a stable property.

However, Williams [16] showed that both *geodesic completeness* and *geodesic incompleteness* may fail to be stable. These two properties are C^0 stable for definite spaces, but for all signatures (s,r) with $s \geq 1$, $r \geq 1$, one can construct examples for which these properties are unstable. (See Del Riego and Dodson [17] for the reasons for these instabilities). However, for Robertson - Walker space-times, geodesic incompleteness can be proved to be stable. See, for example [15].

As mentioned above, global hyperbolicity is a stable property in the set of all time-oriented Lorentz metrics on a fixed manifold. However, A.Garcia Parrado and M. Sanchez [18] have proved that the causal structures of Minkowski and Einstein static space-times remain stable, whereas that of de Sitter space-time become unstable. More precisely, they prove the following theorem :

For any neighbourhood $\mathcal{U}$ in a C^r -Whitney topology, $r = 0, 1, ..., \infty$ of de Sitter space-time, there is a space-time $V \in \mathcal{U}$ such that V is not isocausal to S_1^n . Thus, the causal structure of de Sitter space-time S_1^n is unstable.

Here, isocausality is defined as follows :
Let $\Phi : V_1 \to V_2$ be a global diffeomorphism between two manifolds. We say that the Lorentzian manifold V_2 is causally related to V_1 by Φ, denoted $V_1 \prec_\Phi V_2$, if for every causal future-directed $\overline{u} \in T(V_1), \Phi_* \overline{u} \in T(V_2)$ is causal future directed too. The diffeomorphism Φ is then called a causal mapping. V_2 is said to be causally related to V_1, denoted simply by $V_1 \prec V_2$, if there exists a causal mapping Φ such that $V_1 \prec_\Phi V_2$. Also, two Lorentzian manifolds V_1 and V_2 are called causally equivalent or isocausal if $V_1 \prec V_2$ and $V_2 \prec V_1$. The relation of causal equivalence is denoted by $V_1 \sim V_2$.

In simple language, isocausal space-times have the same causal structure. The authors also prove that there are infinitely many different globally hyperbolic causal structures, and thus different conformal ones on R^2. Another interesting result in this paper is that plane wave solutions with the same number of positive eigenvalues in the frequency matrix have the same causal structure. Thus these solutions have equal causal extensions and causal boundaries.

In a very recent work by Navarro and Minguzzi [19], considering Geroch interval topology, the authors prove that global hyperbolicity is stable in the space-time metrics. They prove that every globally hyperbolic space-time admits a Cauchy hypersurface which remains Cauchy under small perturbations of the space-time metric.

Moreover they prove that if the space-time admits a complete timelike Killing field, then the light cones can be widened preserving both global hyperbolicity and the Killing property of the field. These results about global hyperbolicity are not contradictory because topologies are different. In fact, Geroch's interval topology is one of the coarsest topologies that can be given on the space of (conformal classes of) metrics, and hence the stability in this topology is particularly strong. There is only one other important topology that has been discussed above and which is coarser than Geroch's interval topology: the compact-open topology [1].

In this topology, the metric light cones are bounded only inside a compact set of space-time. Thus a property such as global hyperbolicity need not be stable in this topology.

Specifically, we note that in all these works, the authors have considered the concept of stability referring to a particular topology on the space of space-time metrics. Thus these results concern stability of a property of a space-time considered as a whole. They do not consider the problem of stability of a property under evolution.

Thus, in all these works, no evolution equation such as Einstein's equations, is imposed on the space-time. Such considerations would require a different kind of study like *Linearized stability* [20 - 22] or stability of end states of collapse [23] or proving global existence of solutions of Einstein field equations involving collisionless matter (Rendall [24]). This being an entirely different arena of study, we do not include the discussion of these topics here. The reader is requested to see the references mentioned above.

Finally, we remark that it would be interesting to study stability of space-time properties with respect to the topology developed by Noldus, in the sense that which of the properties are stable and which are not.

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