Jour.Tensor Society **ISSN : 0974-5428**
Vol. 6 No. 2 (2012) 107-112
(S.I. Husain commemorative issue)

A Note on Compact Ricci Solitons

A. Sharfuddin*, Zafar Ahsan and Sharief Deshmukh*****

*Department of Mathematics, Faculty of Natural Sciences
Jamia Millia Islamia, New Delhi-110025, India
E-Mail: sahmad9@jmi.ac.in
**Department of Mathematics, Aligarh Muslim University, Aligarh 202 002, India
E-mail: zafar.ahsan@rediffmail.com
***Department of Mathematics, King Saud University
P.O. Box 2455 Riyadh 11451, Saudi Arabia
E-Mail: shariefd@ksu.edu.sa

Abstract
In this paper we show that an n-dimensional compact Ricci soliton (M, g, ξ, λ), with closed vector field ξ and scalar curvature S satisfying $-\frac{2n}{n-2}\lambda \leq S \leq n\lambda$ or $S \geq (n+2)\lambda - 2(n-1)k_0$, where $(n-1)k_0$ is the infimum of the Ricci curvature, is necessarily an Einstein manifold.

Keywords and phrases : Closed Ricci soliton, Einstien manifold, scalar curvature.
2000 AMS Subject Classification : Primary 53C25.

1. Introduction

A Riemannian manifold (M, g) is said to be a Ricci soliton if there exists a nonzero smooth vector field ξ on M satisfying

$$Ric + \frac{1}{2}\pounds_\xi = \lambda g \tag{1.1}$$

where Ric denotes the Ricci tensor of M, $\pounds_\xi$ denotes the Lie derivative in the direction of ξ and λ is a constant. A Ricci soliton (M, g, ξ, λ) is shrinking soliton, steady soliton or expanding soliton according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. Compact Ricci solitons are the fixed points of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.2}$$

projected from the space of metrics onto its quotient modulo diffeomorphism and scalings (cf. [3]) and the complete Ricci solitons arise as blow-up limits for the Ricci flow on compact manifolds. Topology of Ricci solitons has been studied by Derdzenski, Lopez and Garcia-Rio, Wylie (cf. [4], [7], [10]). If the vector field ξ is gradient ∇f of a smooth function f, the Ricci soliton $(M, g, \nabla f, \lambda)$ is called a gradient Ricci soliton. Gradient Ricci solitons have been studied quite extensively in last decade (cf. [2], [5], [8], [9], [11]). Hamilton [6], conjectured that a compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein and since then the question of obtaining conditions under which a Ricci soliton is an Einstein manifold has been taken up with interest.

In this paper we study closed Ricci solitons, that is, a Ricci soliton (M, g, ξ, λ) with the vector field ξ closed. Clearly a gradient Ricci soliton is closed Ricci soliton. Our efforts are to find conditions under which a compact Ricci soliton is an Einstein manifold. Our main results are the following:

Theorem 1. Let (M, g, ξ, λ) be a compact n-dimensional $(n > 2)$ closed Ricci soliton. If the scalar curvature S of M satisfies the inequality $-\frac{2n}{n-2}\lambda \leq S \leq n\lambda$, then M is an Einstein manifold.

Theorem 2. Let (M, g, ξ, λ), $(\lambda \neq 0)$ be a compact n-dimensional closed Ricci soliton and $k_0 = \frac{1}{n-1} \inf Ric$. If the scalar curvature S of M satisfies the inequality $S \geq (n+2)\lambda - 2(n-1)k_0$, then M is an Einstein manifold.

2. Proof of the Theorem 1.

Since the vector field ξ is closed we define a symmetric (1,1)-tensor ϕ on M by $\phi(X) = \nabla_X \xi$, $X \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection and $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields on M. Then the equation defining Ricci soliton (1.1) takes the form

$$Q + \phi = \lambda I \tag{2.1}$$

where Q is the Ricci operator of M defined by $Ric(X, Y) = g(QX, Y)$, $X, Y \in \mathfrak{X}(M)$. Then the smooth function $f = tr\phi = div\xi$ satisfies

$$\int_M f = 0 \tag{2.2}$$

Also it is straight forward to see that the covariant derivative $(\nabla\phi)(X, Y) = \nabla_X \phi Y - \phi \nabla_X Y$, $\quad X, Y \in \mathfrak{X}(M)$ satisfies

$$R(X, Y)\xi = (\nabla\phi)(X, Y) - (\nabla\phi)(Y, X) \tag{2.3}$$

where R is the curvature tensors of M. As ϕ is symmetric, choosing a local orthonormal frame $\{e_1, .., e_n\}$ on M and using equations (2.1) and (2.3), we deduce the following expression for the gradient ∇f of the function f

$$\nabla f = -Q\xi + \sum_{i=1}^{n} (\nabla \phi)(e_i, e_i) \tag{2.4}$$

It is well known that the gradient ∇S of the scalar curvature S of M satisfies

$$\frac{1}{2}\nabla S = \sum_{i=1}^{n} (\nabla Q)(e_i, e_i) \tag{2.5}$$

Then using $(\nabla Q)(X,Y) = -(\nabla \phi)(X,Y)$ and $f = n\lambda - S$ (which follow from equation (2.1)), in the equations (2.4) and (2.5), we conclude that

$$\nabla f = -\nabla S \text{ and } Q(\xi) = \frac{1}{2}\nabla S \tag{2.6}$$

We use a normal local orthonormal frame $\{e_1, .., e_n\}$ and equations (2.1), (2.5) and (2.6) to compute

$$\begin{aligned} divQ(\xi) &= \sum_{i=1}^{n} g(\nabla_{e_i} Q\xi, e_i) = \sum_{i=1}^{n} e_i g(\xi, Qe_i) \\ &= \sum_{i=1}^{n} [g(\phi e_i, Qe_i) + g(\xi, (\nabla Q)(e_i, e_i)] \\ &= \sum_{i=1}^{n} [g(\phi e_i, \lambda e_i - \phi e_i)] + \frac{1}{2}\xi(S) \\ &= \lambda f - \|\phi\|^2 + \frac{1}{2}div(S\xi) - \frac{1}{2}fS \end{aligned}$$

Using this equation in equation (2.6) we arrive at

$$\frac{1}{2}\Delta S = \lambda f - \|\phi\|^2 + \frac{1}{2}div(S\xi) - \frac{1}{2}fS$$

Integrating this last equation and using (2.2) we get

$$\int_M \left(\|\phi\|^2 + \frac{1}{2}fS \right) = 0 \tag{2.7}$$

As ϕ is symmetric, by Schwartz inequality we have $\|\phi\|^2 \geq \frac{1}{n}f^2$ which together with above integral gives

$$\frac{1}{2n}\int_M f\,(2f + nS) \leq 0 \tag{2.8}$$

Since $n > 2$, using (2.9), the inequality (2.8) takes the form

$$\frac{(n-2)}{2n}\int_M (n\lambda - S)\left(\frac{2n}{n-2}\lambda + S\right) \leq 0$$

If $-\frac{2n}{n-2}\lambda \leq S \leq n\lambda$, above inequality gives either $S = -\frac{2n}{n-2}\lambda$ or else $S = n\lambda$. The equality $S = n\lambda$ together with (2.9) gives $f = 0$ and consequently the integral (2.7) gives $\phi = 0$, which together with (2.5) implies that M is an Einstein manifold with Einstein constant λ. If $S = -\frac{2n}{n-2}\lambda$, then equation (2.9) gives that $f = n\lambda + \frac{2n}{n-2}\lambda =$ a constant. A constant f together with equation (2.2) implies $f = 0$ and as in previous case M is an Einstein manifold.

We also have the following:

Corollary. Let (M, g, ξ, λ) be a compact n-dimensional ($n > 2$) closed Ricci soliton. If the Ricci tensor Ric of M satisfies the inequality

$$Ric(\xi,\xi) \leq \frac{1}{n}f^2.$$

then M is an Einstein manifold.

Proof. Using equation (2.6) we have $Ric(\xi,\xi) = \frac{1}{2}\xi(S) = \frac{1}{2}\left(div(S\xi) - fS\right)$, which on integration together with (2.7) gives

$$\int_M \left(\|\phi\|^2 - Ric(\xi,\xi)\right) = 0 \tag{2.9}$$

This equation could be rearranged as

$$\int_M \left((\|\phi\|^2 - \frac{1}{n}f^2) + (\frac{1}{n}f^2 - Ric(\xi,\xi)\right) = 0 \tag{2.10}$$

The Schwartz inequality for the symmetric tensor ϕ implies $\|\phi\|^2 \geq \frac{1}{n}f^2$ and the equality holding if and only if $\phi = \frac{f}{n}I$. Thus using the condition in the statement of the Corollary in equation (2.10) we get $\phi = \frac{f}{n}I$, which together with equation (2.1) implies that $Ric = (\lambda - \frac{f}{n})g$ and as $n > 2$, we get that $(\lambda - \frac{f}{n})$ is a constant. Since

f is constant in equation (2.2) implies $f = 0$, we get that M is an Einstein manifold.

3. Proof of the Theorem 2

Define a smooth function $h : M \to R$, by $h = \frac{1}{2}\|\xi\|^2$, then using definition of ϕ we immediately get the gradient ∇h of the function h as $\nabla h = \phi(\xi) = \lambda\xi - Q\xi$. Using equation (2.6) we arrive at

$$\nabla(h + \frac{1}{2}S) = \lambda\xi \tag{3.1}$$

which gives

$$\Delta(h + \frac{1}{2}S) = \lambda f$$

Multiplying above equation by $(h + \frac{1}{2}S)$ and using integration by parts, we get

$$-\int_M \left\|\nabla(h + \frac{1}{2}S)\right\|^2 = \lambda \int_M \left(hf + \frac{1}{2}Sf\right)$$

Using equation (3.1) in above equation and $\lambda \neq 0$, we get

$$\int_M \left(\lambda\|\xi\|^2 + hf + \frac{1}{2}Sf\right) = 0 \tag{3.2}$$

Combining equations (2.7) and (2.9) we have

$$\int_M \left(Ric(\xi,\xi) + \frac{1}{2}fS\right) = 0$$

Using above equation and the value $hf = n\lambda h - Sh$ in equation (3.2) we arrive at

$$\int_M \left(\frac{1}{2}((n+2)\lambda - S)\|\xi\|^2 - Ric(\xi,\xi)\right) = 0$$

Using the bound on the Ricci curvature the above equation takes the form of the following inequality

$$\int_M \left(\frac{1}{2}(n+2)\lambda - (n-1)k_0 - \frac{1}{2}S\right)\|\xi\|^2 \geq 0 \tag{3.3}$$

The condition in the statement of the Theorem implies that either $\frac{1}{2}(n+2)\lambda - (n-1)k_0 - \frac{1}{2}S = 0$ or $\xi = 0$. The first implication gives S is a constant and consequently f is constant that is $f = 0$ which proves M is an Einstein manifold. The second implication gives $\phi = 0$ and this proves that M is an Einstein manifold.

References

1. Cao, H. D., Geometry of Ricci solitons, Chinese Ann. Math. Ser. B 27(2006), 121-142

2. Chu, Sun-Chin. Geometry of 3-dimensional gradient Ricci solitons with positive curvature. Comm. Anal. Geom. 13 (2005), no. 1, 129–150.

3. Derdzinski, Compact Ricci solitons. To appear

4. Derdzinski, Andrzej. A Myers-type theorem and compact Ricci solitons. Proc. Amer. Math. Soc. 134 (2006), no. 12, 3645–3648.

5. Eminenti, Manolo; La Nave, Gabriele; Mantegazza, Carlo. Ricci solitons: the equation point of view. Manuscripta Math. 127 (2008), no. 3, 345–367.

6. Hamilton, R. S., The formation of singularities in the Ricci flow, In surveys in differential geometry, Vol. II (Cambridge, MA, 1993). International Press, Cambridge, MA, (1995), 7-136.

7. Lopez, M. F., Garcia-Rio, E., A remark on compact Ricci solitons, Math. Ann. 340(2008), 893-896.

8. Petersen,. P., Wylie, W., Rigidity of gradient Ricci soliton, arXiv:math.DG/0710.3174

9. Petersen, P., Wylie, W., On the classification of gradient Ricci soliton, arXiv:math.DG/0712.1298v4.

10. Wylie, W., Complete shrinking Ricci solitons have finite fundamental group. Proc. Amer. Math. Soc. 136 (2008), no. 5, 1803–1806.

11. Zhang, Xi. Compactness theorems for gradient Ricci solitons. J. Geom. Phys. 56 (2006), no. 12, 2481–2499.