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On Geometry of Lightlike Hypersurfaces of an Indefinite Kenmotsu Manifolds

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Abstract
The geometry of lightlike hypersurface M of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M has been studied. The characterization theorem for distributions to be parallel, Killing, and integrable have been obtained. A necessary and sufficient condition for M to be mixed totally geodesic has also been given.

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1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds as the intersection of normal vector bundle and the tangent bundle is non-trivial. The geometry of lightlike submanifolds were introduced and presented in a book by Duggal and Bejancu [8]. They introduced a non-degenerate screen distribution to construct a non intersecting lightlike transversal vector bundle of the tangent bundle. Many mathematicians have studied lightlike hypersurfaces of semi-Riemannian manifolds (c.f., [2], [4], [5] and [17]).

Physically, lightlike hypersurfaces are interesting in general relativity since they produce models of different types of horizons. For instance, the existence of Killing vector fields has often been used as the most effective symmetry. In fact, since Einstein's field equations are a complicated set of nonlinear partial differential equations, many exact solutions have been found by assuming one or more Killing vector fields (see [1] and [7] for more details and many more references therein). In particular, Carter [6] used this information in the study of a null hypersurface which is also a Killing horizon. Later, the relationship between Killing and geodesic notions were well specified [9].

The contact geometry has significant use in differential equations and phase spaces of dynamical systems ([14] and [16]), whereas the literature about its lightlike case is very limited. Some specific discussion on this matter can be found in [5], [12], [15] and [19]. Lightlike hypersurfaces are also studied in the theory of Electromagnetism (cf. [8], chapter 8). Further, lightlike hypersurfaces of indefinite Sasakian manifolds were studied in [9] and [12]. Some aspect of the geometry of lightlike hypersurfaces of indefinite Kenmotsu manifolds were studied by Massamba in [10] and [11].

In this paper, we study Killing and geodesic lightlike hypersurfaces of indefinite Kenmotsu manifolds and obtain some interesting results.

2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\overline{\phi}, \xi, \eta, \overline{g}\}$, where $\overline{\phi}$ is a (1,1) tensor field, ξ a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\left.\begin{array}{c} \overline{\phi}^2\,\overline{X} = -\overline{X} + \eta(\overline{X})\xi, \quad \eta \circ \overline{\phi} = 0, \quad \overline{\phi}\xi = 0, \quad \eta(\xi) = 1 \\ \overline{g}(\overline{\phi X}, \overline{\phi Y}) = \overline{g}(\overline{X}, \overline{Y}) - \varepsilon\eta(\overline{X})\eta(\overline{Y}), \quad \eta(\overline{X}) = \varepsilon\overline{g}(\overline{X}, \xi), \\ \overline{g}(\xi, \xi) = \varepsilon, \ \ \varepsilon = \pm 1, \end{array}\right\} \tag{2.1}$$

for any $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$, where $\Gamma(T\overline{M})$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [13],

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = g(\overline{\phi X}, \overline{Y})\xi - \varepsilon\eta(\overline{Y})\overline{\phi X}, \qquad \text{and} \ \ \overline{\nabla}_{\overline{X}}\xi = \varepsilon(\overline{X} - \eta(\overline{X})\xi), \tag{2.2}$$

for any $\overline{X}, \overline{Y} \in T\overline{M}$, where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

Let (M, g) be a hypersurface of a $(2n+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ with index s, $0 < s < 2n+1$ and $g = \overline{g}_{|M}$. Then M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $(2n-1)$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [8]. A non-degenerate complementary distribution $S(TM)$ of rank $(2n-1)$ to $TM^{\perp}$ in TM, that is, $TM = TM^{\perp} \perp S(TM)$, is called screen distribution. The following result (cf. [8], Theorem 1.1, page 79) has an important role in studying the geometry of lightlike hypersurfaces.

Theorem A. *Let $(M, g, S(TM))$ be a lightlike hypersurface of $\overline{M}$. Then, there exists a unique vector bundle $tr(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighbourhood $U \subset M$, there exists a unique*

section N of $tr(TM)$ on U satisfying $\overline{g}(N,E)=1$ and $\overline{g}(N,N)=\overline{g}(N,W)=0$, $\forall\, W\in\Gamma(S(TM)|_U)$.
Then, we have the following decomposition:

$$TM=S(TM)\perp TM^{\perp},\qquad T\overline{M}=S(TM)\perp(TM^{\perp}\oplus tr(TM)).\quad (2.3)$$

Throughout this paper, all manifolds are supposed to be paracompact and smooth. We denote by $\Gamma(E)$ the smooth section of the vector bundle E, and by $\perp$ and $\ominus$ the orthogonal and the non-orthogonal direct sum of two vector bundles, respectively.

Let $\overline{\nabla}$, ∇ and ∇^t denote linear connections on $\overline{M}$, M and vector bundle $tr(TM)$, respectively. Then, the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y),\ \ \forall\ X,Y\in\Gamma(TM),\quad (2.4)$$

$$\overline{\nabla}_XV=-A_VX+\nabla^t_XV,\ \ \forall\ V\in\Gamma(tr(TM)),\quad (2.5)$$

where $\{\nabla_XY, A_VX\}$ and $\{h(X,Y),\nabla^t_XV\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_V is the shape operator of M with respect to V. Moreover, in view of decomposition (2.3), equations (2.4) and (2.5) take the form

$$\overline{\nabla}_XY=\nabla_XY+B(X,Y)N\quad (2.6)$$

$$\overline{\nabla}_XN=-A_NX+\tau(X)N\quad (2.7)$$

for any $X,Y\in\Gamma(TM)$ and $N\in\Gamma(tr(TM))$, where $B(X,Y)$ and $\tau(X)$ are local second fundamental form and a 1-form on U, respectively. It follows that

$$B(X,Y)=\overline{g}(\overline{\nabla}_XY,E)=\overline{g}(h(X,Y),E),\quad B(X,E)=0,\ \text{ and}$$

$$\tau(X)=\overline{g}(\nabla^t_XN,E).$$

Let P denote the projection of TM on $S(TM)$ and ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and $TM^{\perp}$, respectively. Then from the decomposition of tangent bundle of lightlike hypersurface, we have

$$\nabla_XPY=\nabla^*_XPY+h^*(X,PY)\quad (2.8)$$

$$\nabla_XE=-A^*_EX+\nabla^{*t}_XE\quad (2.9)$$

for any $X,Y\in\Gamma(TM)$ and $E\in\Gamma(TM^{\perp})$, where h^*, A^* are the second fundamental form and the shape operator of distribution $S(TM)$, respectively.

By direct calculations using Gauss-Weingarten formulae and (2.8) and (2.9), we find

$$g(A_NY,PW)=\overline{g}(N,h^*(Y,PW));\qquad \overline{g}(A_NY,N)=0\quad (2.10)$$

$$g(A^*_EX,PY)=\overline{g}(E,h(X,PY));\qquad \overline{g}(A^*_EX,N)=0\quad (2.11)$$

for any $X, Y, W \in \Gamma(TM)$, $E \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$.

Locally, we define on U

$$C(X, PY) = \overline{g}(h^*(X, PY), N), \text{ and } \lambda(X) = \overline{g}(\nabla_X^{*t} E, N). \tag{2.12}$$

Hence,

$$h^*(X, PY) = C(X, PY)E, \text{ and } \nabla_X^{*t} E = \lambda(X)E. \tag{2.13}$$

On the other hand, by using (2.6), (2.7), (2.9) and (2.12), we obtain

$$\lambda(X) = \overline{g}(\nabla_X E, N) = \overline{g}(\overline{\nabla}_X E, N) = -\overline{g}(E, \overline{\nabla}_X N) = -\tau(X).$$

Thus, locally (2.8) and (2.9) become

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)E, \text{ and } \nabla_X E = -A_E^* X - \tau(X)E. \tag{2.14}$$

Finally, locally (2.10) and (2.11), become

$$g(A_N Y, PW) = C(Y, PW); \qquad \overline{g}(A_N Y, N) = 0, \tag{2.15}$$

$$g(A_E^* X, PY) = B(X, PY); \qquad \overline{g}(A_E^* X, N) = 0. \tag{2.16}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, we have

$$0 = (\overline{\nabla}_X \overline{g})(Y, Z) = X(\overline{g}(Y, Z)) - \overline{g}(\overline{\nabla}_X Y, Z) - \overline{g}(Y, \overline{\nabla}_X Z).$$

By using (2.6) in this equation, we obtain

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \quad \forall X, Y, Z \in \Gamma(S(TM)|_u), \tag{2.17}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) = \overline{g}(N, \cdot)$.

3. Lightlike hypersurfaces of Indefinite Kenmotsu manifolds

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and (M, g) be its lightlike hypersurface tangent to the structure vector field ξ with $\overline{g}(\xi, \xi) = \varepsilon = +1$.

If E is a local section of $TM^{\perp}$, then $\overline{g}(\overline{\phi}E, E) = 0$ implies that $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \bigcap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as vector sub-bundle.

Now, we consider a local section N of $tr(TM)$. Then $\overline{\phi}N$ is tangent to M and belongs to $S(TM)$ as $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$ and $\overline{g}(\overline{\phi}N, N) = 0$.

From (2.1), we have

$$\overline{g}(\overline{\phi}N, \overline{\phi}E) = \overline{g}(N, E) - \eta(N)\eta(E) = \overline{g}(N, E) = 1$$

Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))$ is a direct sum but not orthogonal and is a non degenerate vector subbundle of $S(TM)$ of rank 2.

It is known [5] that if M is tangent to structure vector field ξ, then ξ belongs to $S(TM)$. Since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a non degenerate invariant distribution D_0 of rank $(2n-4)$ on M such that

$$S(TM) = \{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))\} \perp D_0 \perp <\xi> \quad and \quad \overline{\phi}(D_0) = D_0 \tag{3.1}$$

where $<\xi>$ = span ξ.
Moreover, from (2.3) and (3.1), we obtain

$$TM = \{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(tr(TM))\} \perp D_0 \perp <\xi> \perp TM^{\perp}. \tag{3.2}$$

Now, we consider the distributions D and D' on M as follows:

$$D = TM^{\perp} \perp \overline{\phi}(TM^{\perp}) \perp D_0, \quad D' = \overline{\phi}(tr(TM)).$$

Then D is invariant under $\overline{\phi}$ and

$$TM = \{D \oplus D'\} \perp <\xi>. \tag{3.3}$$

If P_1 and Q denote the projection morphisms of TM on D and D', respectively and $U = -\overline{\phi}N, V = -\overline{\phi}E$ are local lightlike vectors, then we write

$$X = P_1X + QX + \eta(X)\xi \tag{3.4}$$

for $X \in \Gamma(TM)$, where $QX = u(X)U$, and u is a differential 1-form locally defined on M by $u(\cdot) = g(V, \cdot)$.
From (3.1) and (3.4), we obtain

$$\overline{\phi}X = \phi X + u(X)N \;\; \text{and} \;\; \phi^2 X = -X + \eta(X)\xi + u(X)U, \;\; \forall X \in \Gamma(TM)$$

where ϕ is a tensor field of type (1, 1) defined on M by $\phi X = \overline{\phi}P_1X$.
Applying ϕ to $\phi^2 X$ and using the fact that $\phi U = 0$, we obtain

$$\phi^3 + \phi = 0$$

which shows that ϕ is an f-structure of constant rank [19].
Using (2.1), we get

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)\nu(X) - u(X)\nu(Y),$$

where ν is a 1-form locally defined on M by $\nu(\cdot) = g(U, \cdot)$.

We have

Lemma 3.1. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then for any $X, Y \in \Gamma(TM)$

$$\nabla_X \xi = X - \eta(X)\xi, \qquad B(X,\xi) = 0, \tag{3.5}$$

$$(\nabla_X u)Y = -B(X, \phi Y) - u(Y)\tau(X) - u(X)\eta(Y), \tag{3.6}$$

and

$$(\nabla_X \phi)Y = u(Y)A_N X - B(X,Y)U + g(\phi X, Y)\xi - \theta(Y)u(X)\xi - \eta(Y)\phi X. \tag{3.7}$$

4. Killing and geodesic lightlike hypersurfaces of indefinite Kenmotsu manifolds
This section is devoted to the study of some geometric aspects of lightlike hypersurface (M, g) of indefinite Kenmotsu manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$, with $\xi \in TM$. We have

Definition 4.1. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$. Then

(a) M is D or $D\perp <\xi>$-totally geodesic (respectively, D'-totally geodesic) if its second fundamental form h satisfies $h(X,Y) = 0$ (equivalently, $B(X,Y) = 0$), for any $X, Y \in \Gamma(D)$ or $\Gamma(D\perp <\xi>)$ (respectively, $X, Y \in \Gamma(D')$);

(b) M is mixed totally geodesic if its second fundamental form h satisfies $h(X,Y) = 0$ (equivalently, $B(X,Y) = 0$), for any $X \in \Gamma(D\perp <\xi>)$ and $Y \in \Gamma(D')$.

It is easy to see that M is mixed totally geodesic between distributions $<\xi>$ and the sub-bundle D' because $B(\xi, U) = 0$.

Proposition 4.2. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field $\xi \in TM$. The Lie derivative with respect to the vector field V is given by

$$(L_V g)(X,Y) = Xu(Y) + Yu(X) + u([X,Y]) - 2g(h(X, \phi Y), E) - 2\eta(Y)u(X), \tag{4.1}$$

for any $X, Y \in TM$.

Proof: A straightforward calculation gives

$$\begin{aligned}
\overline{g}(h(X,\overline{\phi}Y), E) &= \overline{g}(\overline{\nabla}_X \overline{\phi} Y, E) \\
&= X.g(Y,V) - g(Y,[X,V]) - \overline{g}(Y, \overline{\nabla}_V X) \\
&\quad - g(Y, (\overline{\nabla}_X \overline{\phi})E) \\
&= X.g(Y,V) - g(Y,[X,V]) - Vg(Y,X) \\
&\quad + g([V,Y],X) - g(Y,(\overline{\nabla}_X \overline{\phi})E) \\
&= X.u(Y) - (L_V g)(X,Y) + Y.u(X) + u([X,Y]) - g(V, \overline{\nabla}_X Y) \\
&\quad - \eta(Y) g(\overline{\phi} X, E) \\
&= X.u(Y) - (L_V g)(X,Y) + Y.u(X) + u([X,Y]) \\
&\quad - g(E, h(X, \overline{\phi} Y)) - 2\eta(Y)u(X)
\end{aligned}$$

The proof follows from this equation.

Definition 4.3. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$. Then

(a) A distribution Ξ on M is called killing distribution if $(L_X g)(Y, Z) = 0$, for any $X \in \Gamma(\Xi)$ and $Y, Z \in \Gamma(TM)$.

(b) A distribution Ξ on M is called D or $D\perp < \xi >$-killing distribution (respectively, D'-killing distribution) if $(L_X g)(Y, Z) = 0$, for any $X \in \Gamma(\Xi)$ and $Y, Z \in \Gamma(D)$ or $Y, Z \in \Gamma(D\perp < \xi >)$ (respectively, $Y, Z \in \Gamma(D')$).

Lemma 4.4. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then for any $X, Y \in \Gamma(TM)$

$$\overline{g}(h(X, \overline{\phi}Y), E) = u(\nabla_X Y) - \eta(Y)u(X). \tag{4.2}$$

Proof: We have

$$\begin{aligned}\overline{g}(h(X, \overline{\phi}Y), E) &= \overline{g}(\overline{\nabla}_X \overline{\phi}Y, E) = g((\overline{\nabla}_X \overline{\phi})Y, E) + g(\overline{\phi}\overline{\nabla}_X Y, E)\\ &= -\eta(Y)g(\overline{\phi}X, E) + g(\overline{\phi}\overline{\nabla}_X Y, \overline{\phi}E)\\ &= u(\nabla_X Y) - \eta(Y)u(X),\end{aligned}$$

which completes the proof.

Definition 4.5. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$. Then

(a) A vector field W is said to be parallel with respect to the connection ∇ if $\nabla_X W = 0$, for any $X \in \Gamma(TM)$.

(b) A vector field W is D or $< \xi >$ or $D\perp < \xi >$-parallel (respectively, D'-parallel) with respect to the connection ∇ if $\nabla_X W = 0$, for any $X \in \Gamma(D)$ or $< \xi >$ or $X \in \Gamma(D\perp < \xi >)$ (respectively, for any $X \in \Gamma(D')$).

From this definition, we note that a parallel vector field with respect to a connection is not necessarily $D\perp < \xi >$-parallel or D'-parallel vector field. On the other hand, a $D\perp < \xi >$-parallel (D and $< \xi >$-parallel) and D'-parallel vector field is a parallel vector field due to the following relation of a linear connection ∇

$$\begin{aligned}\nabla_{RX+u(X)U+\eta(X)\xi}(\cdot) &= \nabla_{RX+\eta(X)\xi}(\cdot) + u(X)\nabla_U(\cdot), \qquad (4.3)\\ &= \nabla_{RX}(.) + \eta(X)\nabla_\xi(.) + u(X)\nabla_U(.),\end{aligned}$$

for any $X \in \Gamma(TM)$ such that $X = RX + u(X)U + \eta(X)\xi$. Using (3.5), we have

Lemma 4.6. The spacelike vector field ξ is not D'-parallel with respect to the induced connection ∇.

Proposition 4.7. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Suppose that the distribution $D\perp < \xi >$ is parallel with respect to the induced connection ∇ and the vector field V is parallel with respect to the connection $\overline{\nabla}$. Then

$$(L_V g)(X,Y) = 0, \tag{4.4}$$

for any $X \in \Gamma(TM), Y \in \Gamma(D\perp < \xi >)$.

Proof: From (4.1), we get $(L_V g)(X,Y) = Y.u(X) - u([X,Y])$, since $u(Y) = 0$ and $\overline{g}(h(X,\overline{\phi}Y),E) = u(\nabla_X Y) - \eta(Y)u(X) = 0$, for $X \in \Gamma(TM), Y \in \Gamma(D\perp <\xi >)$. Now since V is parallel with respect to the connection $\overline{\nabla}$, we have

$$\begin{aligned}0 &= \overline{g}(\overline{\nabla}_Y V, X) = Y.\overline{g}(V,X) - \overline{g}(V, \overline{\nabla}_Y X)\\ &= Y.\overline{g}(V,X) - \overline{g}(V,[Y,X]) - \overline{g}(V, \nabla_X Y)\\ &= Y.\overline{g}(V,X) - \overline{g}(V,[V,X]) = Y.u(Y) - u([X,Y]).\end{aligned}$$

whereby the assertion is complete.

It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Therefore, we investigate the integrability of screen distribution.

Proposition 4.8. Let $\overline{M}$ be an indefinite Kenmotsu manifold and M be a $D\perp < \xi >$ totally geodesic lightlike hypersurface of $\overline{M}$ with $\xi \in TM$. Then

$$(L_V g)(X,Y) = -(L_V g)(Y,X), \tag{4.5}$$

for any $X, Y \in \Gamma(D\perp < \xi >)$. Moreover, the distribution $D\perp < \xi >$ is integrable if and only if $\overline{\phi}(TM^{\perp})$ is a $D\perp < \xi >$-Killing distribution on M.

Proof: Since M is a $D\perp < \xi >$ totally geodesic lightlike hypersurface of $\overline{M}$, using equation (4.1), we get $(L_V g)(X,Y) = u([X,Y])$ for any $X, Y \in \Gamma(D\perp < \xi >)$, whereby equation (4.5) follows. Now, if distribution $D\perp < \xi >$ is integrable then $g([X,Y],V) = 0$, for every $X, Y \in D\perp < \xi >$. Since $(L_V g)(X,Y) = u([X,Y])$, we get the result.

Theorem 4.9. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then the distribution $D\perp < \xi >$ is integrable if and only if

$$B(\phi X, Y) = B(X, \phi Y), \tag{4.6}$$

for any $X, Y \in \Gamma(D\perp < \xi >)$.

Proof: We have

$$\begin{aligned}\overline{g}([X,Y],\overline{\phi}E) &= \overline{g}(\overline{\nabla}_X Y, \overline{\phi}E) - \overline{g}(\overline{\nabla}_Y X, \overline{\phi}E)\\ &= \overline{g}(\overline{\nabla}_Y \overline{\phi}X, E) - \overline{g}(\overline{\nabla}_X \overline{\phi}Y, E) - \overline{g}((\overline{\nabla}_Y \overline{\phi})X, E) + \overline{g}((\overline{\nabla}_X \overline{\phi})Y, E)\\ &= \overline{g}(h(\overline{\phi}X, Y) - h(X, \overline{\phi}Y), E) = B(\phi X, Y) - B(X, \phi Y).\end{aligned}$$

The assertion follows from this equation.

Theorem 4.10. Let $\overline{M}$ be an indefinite Kenmotsu manifold and M be a $D\perp < \xi >$-totally geodesic lightlike hypersurface of $\overline{M}$ with $\xi \in TM$. Then $\overline{\phi}(TM^{\perp})$ is a $D\perp < \xi >$-Killing distribution.

Proof: Using equation (4.1) and Theorem 4.9, we have

$$(L_V g)(X,Y) - (L_V g)(Y,X) = 2(u([X,Y]) - \overline{g}(h(X,\overline{\phi}Y) - h(\overline{\phi}X,Y),E) = 0,$$

for any $X, Y \in \Gamma(D\perp < \xi >)$.

From this equation and equation (4.5), we obtain $(L_V g)(X,Y) = (L_V g)(Y,X) = -(L_V g)(X,Y)$, that is $(L_V g)(X,Y) = 0$. Hence, $\overline{\phi}(TM^{\perp})$ is a $D\perp < \xi >$-killing distribution on M.

It is well known that the second fundamental form and the shape operator of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. In the case of lightlike hypersurface, it is observed from (2.8) and (2.9) that the second fundamental form on M and their screen distribution $S(TM)$ are related to their respective shape operators A_N and A^*_E. As the shape operator is an information tool in studying the geometry of submanifolds, we study these operators and give their implications in lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$.

We have

Theorem 4.11 Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then

$$B(X,U) = C(X,V) = u(A_N X), \tag{4.7}$$

$$B(X,V) = u(A^*_E X), \tag{4.8}$$

and

$$\nabla^*_X V = \phi(A^*_E X) - C(X,V)E - \tau(X)V - u(X)\xi. \tag{4.9}$$

for any $X \in \Gamma(TM)$.

Proof: The relation $B(X,U) = C(X,V)$ is trivial. Using the definition of B and the fact that $\overline{\nabla}$ is a Levi-Civita connection, we get

$$C(X,V) = g(A_N X, V) = u(A_N X),$$

which gives equation (4.7).

Also, we have

$$\begin{aligned} B(X,V)N &= \overline{\nabla}_X V - \nabla_X V = -\overline{\nabla}_X \overline{\phi} E + \nabla_X \overline{\phi} E, \\ &= -(\overline{\nabla}_X \overline{\phi})E - \overline{\phi}(\overline{\nabla}_X E) + \nabla^*_X \overline{\phi} E + C(X,\overline{\phi}E)E \\ &= -\overline{g}(\overline{\phi}X,E)\xi + \overline{\phi}(A^*_E X) + \tau(X)\overline{\phi}E + \nabla^*_X \overline{\phi}E + C(X,\overline{\phi}E)E \end{aligned}$$

or,

$$(B(X,V)-u(A^*_E X))N = -u(X)\xi+\phi(A^*_E X)-\tau(X)V+\nabla^*_X\overline{\phi}E+C(X,\overline{\phi}E)E.$$

Comparing the components of $\Gamma(TM)$ and $\Gamma(tr(TM))$ in the above relation, we obtain equations (4.8) and (4.9).

Theorem 4.12. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then, M is mixed totally geodesic if and only if

$$A_N X \in \Gamma(\overline{\phi}(TM^{\perp})\perp D_0\perp <\xi>), \tag{4.10}$$

for $X \in \Gamma(D\perp <\xi>)$.

Proof: From definition 4.1 (b) and equation (4.7), we obtain $u(A_N X) = \overline{g}(A_N X, V) = 0$. Moreover, $(A_N X, N) = 0$. Hence $A_N X \in \Gamma(\overline{\phi}(TM^{\perp})\perp D_0\perp <\xi>)$.

Theorem 4.13 Let M be a lightlike hypersurface of an indefinite Kenmostu manifold $\overline{M}$ with $\xi \in TM$. Then M is $\Gamma(D\perp <\xi>)$-totally geodesic if and only if

$$A^*_E X \in \Gamma(\overline{\phi}(TM^{\perp})), \tag{4.11}$$

for any $X \in \Gamma(D\perp <\xi>)$.

Proof: Assum that M is $(D\perp <\xi>)$-totally geodesic. Then, from equation (4.8), we obtain $\overline{g}(A^*_E X, V) = 0$, for $X \in \Gamma(D\perp <\xi>)$. Also, $g(A^*_E X, N) = 0$. Hence, $A^*_E X \in \Gamma(\overline{\phi}(TM^{\perp})\perp D_0\perp <\xi>)$. Now, if $A^*_E X \in \Gamma(D_0\perp <\xi>)$ and given that $D_0\perp <\xi>$ is non-degenerate, there exists $Z \in \Gamma(D_0\perp <\xi>)$ such that $\overline{g}(A^*_E X, Z) \neq 0$. But from (2.6) and (2.14), we obtain

$$\begin{aligned} g(A^*_E X, Z) &= -\overline{g}(\overline{\nabla}_X E, Z) = -X.\overline{g}(E,Z) + \overline{g}(E, \overline{\nabla}_X Z) \\ &= \overline{g}(E, \nabla_X Z) + B(X,Z)\overline{g}(E,N) = 0. \end{aligned}$$

Thus, $A^*_E X \in \Gamma(\overline{\phi}(TM^{\perp}))$ and is not an element of $\Gamma(D_0\perp <\xi>)$.

Conversely, suppose that for any $X \in \Gamma(D\perp <\xi>)$, $A^*_E X \in \Gamma(\overline{\phi}(TM^{\perp}))$. Let $B_{D\perp<\xi>} = \{E, \overline{\phi}E, \xi, F_i\}, i = 1,2,...,2n-4$, be a local orthonormal field of frames of $D\perp <\xi>)$ such that $D_0 = span\{F_i\}$. Now, we want to show that $B(X,.)$ vanishes in each element of $B_{D\perp<\xi>}$. For any $X \in \Gamma(D\perp <\xi>)$, $u(A^*_E X) = 0 = B(X,V)$, where V is defined as $-\overline{\phi}E$. Also from second relation of equation (2.2), we get $B(X,\xi) = \overline{g}(\overline{\nabla}_X\xi, E) = \overline{g}(X-\eta(X)\xi, E) = 0$. Next, $B(X,F_i) = \overline{g}(\overline{\nabla}_X F_i, E) = -\overline{g}(F_i, \nabla_X E) = \overline{g}(F_i, A^*_E X) = 0$, since $D_0\perp\overline{\phi}(TM^{\perp})$. Let Y be an element of $\Gamma(D\perp <\xi>)$. Locally, we have $Y = \theta(Y)E + v(Y)V + \eta(Y)\xi + \sum_i \frac{\overline{g}(Y,F_i)}{\overline{g}(F_i,F_i)}F_i \in \Gamma(D\perp <\xi>)$, with $\overline{g}(F_i,F_i) \neq 0$ because of the non-degeneracy of D_0. So, we have $B(X,Y) = \theta(Y)B(X,E) + v(Y)B(X,V) + \eta(Y)B(X,\xi) + \sum_i \frac{\overline{g}(Y,F_i)}{\overline{g}(F_i,F_i)}B(X,F_i) = 0$. Hence M is $D\perp <\xi>$ totally geodesic.

The expression of the shape operators A_N and A^*_E can be computed explicitly in the following way:

According to the decomposition (3.2), we consider a local field of frames on M i.e.

$$\{\overline{\phi}E, \overline{\phi}N, \xi, E, F_i\}_{1\leq i\leq 2n-4}, \tag{4.12}$$

on $u \subset M$; where $\{F_i\}_{1\leq i\leq 2n-4}$ is an orthonormal field of frames of D_0.

Lemma 4.14. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then

$$A_N X = \sum_{i=1}^{2n-4} \frac{C(X, F_i)}{g(F_i, F_i)} F_i + C(X, \xi)\xi + C(X, U)V + C(X, V)U, \tag{4.13}$$

and

$$A^*_E X = \sum_{i=1}^{2n-4} \frac{B(X, F_i)}{g(F_i, F_i)} F_i + B(X, \xi)\xi + B(X, U)V + B(X, V)U, \tag{4.14}$$

for any $X \in \Gamma(TM)$ and U, V are any local lightlike vector fields.

Proof: Using (4.12), we have

$$A_N X = \sum_{i=1}^{2n-4} \lambda_i F_i + \gamma\xi + \delta E + \alpha\overline{\phi}E + \beta\overline{\phi}N.$$

From (2.7) and (2.14), we obtain $\lambda_i g(F_i, F_i) = g(A_N X, F_i) = C(X, F_i)$. Since D_0 is a non-degenerate distribution on M, $g(F_i, F_i) \neq 0$ and we have $\lambda_i = \frac{C(X,F_i)}{g(F_i,F_i)}$ and $\gamma = g(A_N X, \xi) = \eta(A_N X) = C(X, \xi)$, $\delta = g(A_N X, N) = 0$, $\alpha = -g(A_N X, U) = -C(X, U)$, $\beta = -g(A_N X, V) = -C(X, V)$ which proves (4.13). Similarly, we obtain (4.14).

Theorem 4.15. Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. Then, M is $D\perp < \xi >$-totally geodesic if and only if for any $X \in \Gamma(D\perp < \xi >)$,

$$A^*_E X = u(A_N X)V. \tag{4.15}$$

The proof follows from Theorem 4.13 and equation (4.14).

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