

Some Results on Trans-Sasakian Manifolds admitting Semi-symmetric Non Metric Connection

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(Dedicated to Prof. K. S. Amur on his 80th birth year)

Abstract

In the present paper we obtained several results on trans-sasakian manifold, admitting semi-symmetric non metric connection under the condition, $\phi(\text{grad } \alpha) = (n - 2) \text{grad } \beta$.

Keywords and Phrases : Trans-sasakian manifolds, semi-symmetric non metric connection.

1. Introduction

In 1924, Friedman and Schouter [1] introduced the notion of semi-symmetric linear connection on differentiable manifold. Then in 1932, Hyden [2] introduced the idea of metric connection with a torsion on a Riemannian manifold. A systematic study of semi symmetric metric connection on Riemannian manifold has been given by K. Yano [3] in 1970. In 1992, Agashe and Chaffe [7] have defined the notion of semi-symmetric non-metric connection on Riemannian manifold. In 2002 S. K. Srivastva studied the semi symmetric non metric connection on manifold with generalized structures [12]. In 2007 C. S. Bagwadi [10] has obtained some results on trans-sasakian manifold with respect to semi-symmetric metric connection.

2. Preliminaries

An odd dimensional differentiable manifold M^n is said to be an almost contact metric manifold [10] if it admits a $a(1, 1)$ tensor field ϕ , a vector field ξ ,

a 1-form η and a Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad (1)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \quad (3)$$

for all vector field $X, Y \in M^n$.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be a trans-sasakian structure of the type (α, β) [10] if

$$(\nabla_X \phi)(Y) = \alpha[g(X, Y)\xi - \eta(Y)X] + \beta[g(X, Y) - \eta(Y)\phi X] \quad (4)$$

for some smooth functions α and β on M^n .

Let M^n be a n-dimensional trans-sasakian manifold, then from equation (4) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \quad (5)$$

and

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \quad (6)$$

In n-dimensional trans-sasakian manifold, we have

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X) \quad (7)$$

$$2\alpha\beta + \xi\alpha = 0 \quad (8)$$

and

$$S(X, \xi) = [(n-1)(\alpha^2 - \beta^2) - \xi\beta]\eta(X) - (n-2)X\beta - \phi(X)\alpha. \quad (9)$$

Further, in trans-sasakian manifold of type (α, β) , we have

$$\phi(\text{grad } \alpha) = (n-2)\text{grad } \beta \quad (10)$$

Using (10), the equations (7) and (9) reduces to

$$R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X) \quad (11)$$

and

$$S(X, \xi) = (n-1)(\alpha^2 - \beta^2)\eta(X) \quad (12)$$

Let (M^n, g) be a n-dimensional Riemannian manifold with metric tensor g and ∇ be the Riemannian connection (Levi-civita connection) on M^n . A linear

connection $\bar{\nabla}$ on (M^n, g) is said to be semi-symmetric non metric if the torsion tensor T of the connection for trans-sasakian manifold satisfying

$$TR(X, Y) = \eta(Y)X - \eta(X)Y \quad (13)$$

where η is 1-form on M^n with ξ as a associated vector field such that

$$\eta(X) = g(X, \xi)$$

and

$$(\bar{\nabla}_X g)(Y, Z) = -\eta(Y)g(X, Z) - \eta(Z)g(X, Y) \quad (14)$$

The relation between semi-symmetric non-metric connection $\bar{\nabla}$ and Riemannian connection ∇ of (M^n, g) has been obtained and given by [7]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X \quad (15)$$

Further, a relation between curvature tensor R and \bar{R} of ∇ and $\bar{\nabla}$ are given by

$$\bar{R}(X, Y)Z = R(X, Y)Z - K(Y, Z)X + K(X, Z)Y \quad (16)$$

where $K(X, Y)$ is tensor field of type $(0, 2)$ given by

$$K(X, Y) = (\nabla_X \eta)Y - \eta(X)\eta(Y) \quad (17)$$

For a trans-sasakian manifold M^n under the condition (10) with semi-symmetric metric connection, we have following useful results:

$$(i) \quad K(\nabla_Y \xi, Z) = -\alpha^2[g(Y, Z) - \eta(Y)\eta(Z)] - 2\alpha\beta g(\phi Y, Z) + \beta^2[g(Y, Z) - \eta(Y)\eta(Z)] \quad (17.1)$$

$$(ii) \quad K(Y, \nabla_Z \xi) = \alpha^2[g(Y, Z) - \eta(Y)\eta(Z)] + \beta^2[g(Y, Z) - \eta(Y)\eta(Z)]. \quad (17.2)$$

Theorem 1. For a trans-sasakian manifold M^n under the condition (10) with semi symmetric non-metric connection, the following results are true

$$K(Y, Z) = \alpha g(Y, \phi Z) + \beta g(Y, Z) - (\beta + 1)\eta(Y)\eta(Z), \quad (18.1)$$

$$(K(Y, \xi) = -\eta(Y), \quad (18.2)$$

$$K(\bar{\nabla}_Y \xi, Z) = (\beta^2 - \alpha^2 - \beta)g(Y, Z) - \alpha(2\beta + 1)g(\phi Y, Z) + (\alpha^2 - \beta^2 - \beta - 1)\eta(Y)\eta(Z), \quad (18.3)$$

$$K(Y, \bar{\nabla}_Z \xi) = (\alpha^2 + \beta^2 + \beta)g(Y, Z) - \alpha g(\phi Y, Z) - (\alpha^2 + \beta^2 + \beta + 1)\eta(Y)\eta(Z). \quad (18.4)$$

Proof. (18.1). We have

$$K(Y, Z) = (\bar{\nabla}_Y \eta)Z - \eta(Y)\eta(Z).$$

Using equations (2), (3), (6) and simplifying, we get

$$K(Y, Z) = \alpha g(Y, \phi Z) + \beta g(Y, Z) - (\beta + 1)\eta(Y)\eta(Z).$$

Proof. (18.2). Putting $Z = \xi$ in equation (18.1), we have

$$K(Y, \xi) = \alpha g(Y, \phi \xi) + \beta g(Y, \xi) - (\beta + 1)\eta(Y)\eta(\xi).$$

Using equation(1) and (3), we obtained

$$K(Y, \xi) = -\eta(y).$$

Proof. (18.3). Since

$$\begin{aligned}\bar{\nabla}_X Y &= \nabla_X Y + \eta(Y)X \\ K(\bar{\nabla}_Y \xi, Z) &= K(\nabla_Y \xi + \eta(\xi)Y, Z) \\ &= K(\nabla_Y \xi + Y, Z).\end{aligned}$$

Using equation (18.1) this gives

$$\begin{aligned}K(\bar{\nabla}_Y \xi, Z) &= \alpha g(\nabla_Y \xi + Y, \phi Z) + \beta g(\nabla_Y \xi + Y, Z) - (\beta + 1)\eta(\nabla_Y \xi + Y)\eta(Z) \\ &= \alpha g(\nabla_Y \xi, \phi Z) + \alpha g(Y, \phi Z) + \beta g(\nabla_Y \xi, Z) + \beta g(Y, Z) \\ &\quad - (\beta + 1)\eta(\nabla_Y \xi)\eta(Z) - (\beta + 1)\eta(Y)\eta(Z)\end{aligned}$$

Using equation (5) and simplify we get

$$\begin{aligned}K(\bar{\nabla}_Y \xi, Z) &= (\beta^2 - \alpha^2 - \beta)g(Y, Z) - \alpha(2\beta + 1)g(\phi Y, Z) \\ &\quad + (\alpha^2 - \beta^2 - \beta - 1)\eta(Y)\eta(Z)\end{aligned}$$

Proof. (18.4).

$$\begin{aligned}K(Y, \bar{\nabla}_Z \xi) &= K(Y, \nabla_Z \xi + \eta(\xi)Z) \\ &= K(Y, \nabla_Z \xi + Z).\end{aligned}$$

Using equations (5),(18.1) and (18.3), we get the required result.

Theorem 2. For a trans-sasakian manifold M^n under the condition (10) with semi symmetric non metric connection we have

$$\begin{aligned}(\bar{\nabla}_\xi K)(Y, Z) - (\bar{\nabla}_Y K)(\xi, Z) &= (2\beta - \alpha^2 + \beta^2)g(Y, Z) \\ &\quad - 2\alpha(\beta + 1)g(\phi Y, Z) + (\alpha^2 - \beta^2 - 3\beta)\eta(Y)\eta(Z)\end{aligned}\tag{19}$$

Proof. Since,

$$L_\xi g(Y, Z) = 2\beta[g(Y, Z) - \eta(Y)\eta(Z)]\tag{20}$$

for all X and Y , where L is Lie derivative.

From equation (20) and $k(Y, Z) = g(AY, Z)$, we have

$$(L_\xi K)(Y, Z) = 2\beta K(Y, Z) + \beta\eta(Y)\eta(Z). \quad (21)$$

Now,

$$\begin{aligned} (\bar{\nabla}_\xi K)(Y, Z) &= \xi K(Y, Z) - K(\bar{\nabla}_\xi Y, Z) - K(Y, \bar{\nabla}_\xi Z) \\ &= \xi K(Y, Z) - K(\nabla_\xi Y + \eta(Y)\xi, Z) - K(Y, \nabla_\xi Z + \eta(Z)\xi) \\ &= \xi K(Y, Z) - K([\xi, Y] + \nabla_Y \xi, Z) - \eta(Y)K(\xi, Z) \\ &\quad - K(Y, [\xi, Z] + \nabla_Z \xi) - \eta(Z)K(Y, \xi) \\ &= (L_\xi K)(Y, Z) - K(\nabla_Y \xi, Z) - K(Y, \nabla_Z \xi) + 2\eta(Y)\eta(Z). \end{aligned}$$

Using equation (17.1), (17.2), (18.1) and (21) we get

$$(\bar{\nabla}_\xi K)(Y, Z) = -(\beta - 2)\eta(Y)\eta(Z). \quad (22)$$

Again,

$$\begin{aligned} (\bar{\nabla}_Y K)(\xi, Z) &= YK(\xi, Z) - K(\bar{\nabla}_Y \xi, Z) - K(\xi, \bar{\nabla}_Y Z) \\ &= -Y\eta(Z) - K(\bar{\nabla}_Y \xi, Z) - K(\xi, \nabla_Y Z + \eta(Z)Y) \\ &= -Y\eta(Z) - K(\bar{\nabla}_Y \xi, Z) + \eta(\nabla_Y Z) + \eta(Y)\eta(Z) \\ &= -[Y\eta(Z) - \eta(\nabla_Y Z)] - K(\bar{\nabla}_Y \xi, Z) + \eta(Y)\eta(Z) \\ &= -(\nabla_Y \eta)(Z) - K(\bar{\nabla}_Y \xi, Z) + \eta(Y)\eta(Z). \end{aligned}$$

Using equations (6), (18.3) and simplify, we have

$$\begin{aligned} (\bar{\nabla}_Y K)(\xi, Z) &= 2\alpha g(\phi Y, Z) - 2\beta g(Y, Z) + \alpha^2[g(Y, Z) - \eta(Y)\eta(Z)] \\ &\quad + 2\alpha\beta g(\phi Y, Z) - \beta^2[g(Y, Z) - \eta(Y)\eta(Z)] + 2(\beta + 1)\eta(Y)\eta(Z). \end{aligned} \quad (23)$$

Subtracting equation (23) from (22), we get

$$\begin{aligned} (\bar{\nabla}_\xi K)(Y, Z) - (\bar{\nabla}_Y K)(\xi, Z) &= (2\beta - \alpha^2 + \beta^2)g(Y, Z) \\ &\quad - 2\alpha(\beta + 1)g(\phi Y, Z) + (\alpha^2 - \beta^2 - 3\beta)\eta(Y)\eta(Z). \end{aligned}$$

Theorem 3. For a trans-Sasakian manifold $M^n(n > 1)$ under the condition (10), we have

$$\begin{aligned} (\bar{\nabla}_\xi S)(Y, Z) - (\bar{\nabla}_Y S)(\xi, Z) &= (n - 1)\alpha(\alpha^2 - \beta^2)g(\phi Y, Z) \\ &\quad - \beta(n - 1)(\alpha^2 - \beta^2)g(Y, Z) - \alpha S(\phi Y, Z) + (\beta + 1)S(Y, Z) \\ &\quad - (n - 1)(\alpha^2 - \beta^2)\eta(Y)\eta(Z) \end{aligned} \quad (24)$$

Proof. For a symmetric endomorphism Q of the tangent space at a point of M , we express the Ricci tensor as

$$S(Y, Z) = g(QY, Z). \quad (25)$$

In trans-Sasakian manifold, from equations (20) and (25), we have

$$(L_\xi S)(Y, Z) = 2\beta S(Y, Z) - 2\beta(n-1)(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \quad (26)$$

Now,

$$\begin{aligned} (\bar{\nabla}_\xi S)(Y, Z) &= \xi S(Y, Z) - S(\bar{\nabla}_\xi Y, Z) - S(Y, \bar{\nabla}_\xi Z) \\ &= \xi S(Y, Z) - S(\nabla_\xi Y + \eta(Y)\xi, Z) - S(Y, \nabla_\xi Z + \eta(Z)\xi) \\ &= \xi S(Y, Z) - S([\xi, Y] + \nabla_Y \xi, Z) - \eta(Y)S(\xi, Z) \\ &\quad - S(Y, [\xi, Z] + \nabla_Z \xi) - \eta(Z)S(Y, \xi) \\ &= (L_\xi S)(Y, Z) - S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) - \eta(Y)S(\xi, Z) \\ &\quad - \eta(Z)S(Y, \xi). \end{aligned}$$

Using equations (5) and (26) we get

$$(\bar{\nabla}_\xi S)(Y, Z) = -2(n-1)(\alpha^2 - \beta^2)\eta(Y)\eta(Z) \quad (27)$$

Again,

$$\begin{aligned} (\bar{\nabla}_Y S)(\xi, Z) &= Y S(\xi, Z) - S(\bar{\nabla}_Y \xi, Z) - S(\xi, \bar{\nabla}_Y Z) \\ &= Y S(\xi, Z) - S(\nabla_Y \xi + \eta(\xi)Y, Z) - S(\xi, \nabla_Y Z + \eta(Z)Y) \\ &= Y S(\xi, Z) - S(\nabla_Y \xi, Z) - S(Y, Z) - S(\xi, \nabla_Y Z) - \eta(Z)S(\xi, Y). \end{aligned}$$

Using equations (5) and (12)

$$\begin{aligned} (\bar{\nabla}_Y S)(\xi, Z) &= (n-1)(\alpha^2 - \beta^2)(\nabla_Y \eta)(Z) + \alpha S(\phi Y, Z) - (\beta+1)S(Y, Z) \\ &\quad + \beta \eta(Y)S(\xi, Z) - (n-1)(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \end{aligned}$$

Using (6) and (12), again we get

$$\begin{aligned} (\bar{\nabla}_Y S)(\xi, Z) &= -(n-1)(\alpha^2 - \beta^2)\alpha g(\phi Y, Z) \\ &\quad + \beta(n-1)(\alpha^2 - \beta^2)g(Y, Z) + \alpha S(\phi Y, Z) - (\beta+1)S(Y, Z) \\ &\quad - (n-1)(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \end{aligned} \quad (28)$$

Subtracting equation(28) from (27) we get

$$\begin{aligned} (\bar{\nabla}_\xi S)(Y, Z) - (\bar{\nabla}_Y S)(\xi, Z) &= (n-1)\alpha(\alpha^2 - \beta^2)g(\phi Y, Z) \\ &\quad - \beta(n-1)(\alpha^2 - \beta^2)g(Y, Z) - \alpha S(\phi Y, Z) + (\beta+1)S(Y, Z) \\ &\quad - (n-1)(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \end{aligned}$$

Definition. A trans-Sasakian manifold M^n is said to be η -Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for any vector field X, Y where a and b are smooth function on M^n .

Theorem 4. Suppose a trans-Sasakian manifold M^n ($n > 2$) under the condition (10) admits a semi-symmetric non metric connection whose curvature tensor with respect to this connection is conservative, then the manifold is η -Einstein.

Proof. Using equations (16), (19), (24) and simplify we get

$$S(Y, Z) = f_1(\alpha, \beta)g(Y, Z) + f_2(\alpha, \beta)\eta(Y)\eta(Z)$$

which is a η -Einstein manifold.

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