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On the Lanczos Potential for Petrov Type III Spacetimes

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Abstract
The Lanczos potential for some well known Petrov type III metrics have been obtained.

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1. Introduction

In the recent years there has been much interest in the study of Lanczos potential ([1-13], [16-20]). This potential is a rank three tensor L_{ijk} and satisfies the following symmetries

$$(40 \text{ conditions}) \; L_{ijk} = -L_{jik} \tag{1}$$

$$(4 \text{ conditions}) \; L_i{}^t{}_t = 0 \; (\text{or}, g^{kl} L_{kil} = 0) \tag{2}$$

$$(4 \text{ conditions}) \; L_{ijk} + L_{jki} + L_{kij} = 0 \; (\text{or}, {}^*L_i{}^t{}_t = 0) \tag{3}$$

In this way the tensor field L_{ijk}, which has sixtyfour independent components, has been reduced to sixteen independent components. In order to have a perfect match with the Weyl tensor, Lanczos imposed six equations

$$L^k_{ij;k} = 0 \tag{4}$$

so that L_{ijk} is a field with only ten effective degrees of freedom. Equation (4) is known as Lanczos differential guage condition. Lanczos [23] originally created the gravitational field through the equation

$$C_{hijk} = L_{[hi][j;k]} + L_{[jk][h;i]} - {}^*L^*{}_{[hi][j;k]} - {}^*L^*{}_{[jk][h;i]} \tag{5}$$

where dual operation is applied to each pair of antisymmetric brackets as indicated and the double dual is defined as ${}^*A^*{}_{hijk} = \frac{1}{4}\eta_{hilm}\eta_{jkno}A^{lmno}$.

Equation (5) is now known as Weyl-Lanczos equation and has been written by Dolan and Kim [12] in a more convenient form as

$$C_{hijk} = L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + L_{(hk)}g_{ij} + L_{(ij)}g_{hk} - L_{(hj)}g_{ik} - L_{(ik)}g_{hj}$$

$$+\frac{2}{3}L^{pq}{}_{p;q}(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{6}$$

where

$$L_{hk} = L^t{}_{hk;t} - L^t{}_{ht;k} \tag{7}$$

and round bracket denotes symmetrization.

From equations (2), (4) and (7), the Weyl-Lanczos equation (6) can also be expressed as

$$C_{hijk} = L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + \frac{1}{2}(L^p{}_{hk;p} + L^p{}_{kh;p})g_{ij} + \frac{1}{2}(L^p{}_{ij;p} + L^p{}_{ji;p})g_{hk}$$

$$-\frac{1}{2}(L^p{}_{hj;p} + L^p{}_{jh;p})g_{ik} - \frac{1}{2}(L^p{}_{ik;p} + L^p{}_{ki;p})g_{hj} \tag{8}$$

For a given geometry the construction of Lanczos potential L_{ijk} is equivalent to solving equation (6)/(8) with equations (2)-(4) as constraints. There are several methods of solving equation (6)/(8) although none of them are straight forward as one would like them to be (cf.,[2-7], [9-12]). In this paper, we shall obtain the Lanczos potentials of some well known type III metrics using the scheme given by Ares de Parga, Chavoya and Lopez-Bonilla [10].

2. Type III metrics and NP formalism

With a choice of NP-null tetrad $\{l^i, n^i, m^i, \bar{m}^i\}$, the sixteen real independent components of the Lanczos potential L_{ijk} are given by the eight complex quantities $L_i, i = 0, 1, 2, ..., 7$, known as Lanczos scalars. These scalars are given by

$$L_0 = L_{ijk} l^i m^j l^k \ , \ L_1 = L_{ijk} l^i m^j \bar{m}^k$$
$$L_2 = L_{ijk} \bar{m}^i n^j l^k \ , \ L_3 = L_{ijk} \bar{m}^i n^j \bar{m}^k$$
$$L_4 = L_{ijk} l^i m^j m^k \ , \ L_5 = L_{ijk} l^i m^j n^k \tag{9}$$
$$L_6 = L_{ijk} \bar{m}^i n^j m^k \ , \ L_7 = L_{ijk} \bar{m}^i n^j n^k$$

Therefore, once these Lanczos scalars are known, we can obtain the Lanczos potential through the completness relation ([3],[10])

$$L_{ijk} = K_{ijk} + \bar{K}_{ijk} \tag{10}$$

between the Lanczos spin tensor and the Lanczos scalars $L_i (i = 0, 1, 2, ..., 7)$ where

$$K_{ijk} = L_0 U_{ij} n_k + L_1 (M_{ij} n_k - U_{ij} m_k) + L_2 (V_{ij} n_k - M_{ij} m_k) - L_3 V_{ij} m_k$$
$$-L_4 U_{ij} \bar{m}_k + L_5 (U_{ij} l_k - M_{ij} \bar{m}_k) + L_6 (M_{ij} l_k - V_{ij} \bar{m}_k) + L_7 V_{ij} l_k \tag{11}$$

and

$$M_{ij} = l_i n_j - l_j n_i + m_i \bar{m}_j - m_j \bar{m}_i$$
$$U_{ij} = -n_i \bar{m}_j + n_j \bar{m}_i \ , \ V_{ij} = l_i m_j - l_j m_i \tag{12}$$

Petrov type III regions are associated with a kind of longitudinal gravitational radiation. In such regions, the tidal forces have a shearing effect. Petrov type III radiation decay is proportional to $\frac{1}{r^2}$. Due to such importance of type III fields, we shall obtain the Lanczos potential of some well known type III metrics following the techniques given in [10]. We shall write the null tetrad for each metric under consideration along with other necessary informations such as non-zero spin coefficients, intrinsic derivatives etc. If L_i $(i = 0, 1, 2, ...7)$ are known then the Lanczos potential can be obtained from equation (10) which in turn generates the gravitational field through equation (6).

(i) Kaigorodov metric

The Kaigorodov space is a homogeneous Einstein space whose metric in (x, y, v, u) is given by [22]

$$ds^2 = 2(kx)^{-2}(dx^2 + dy^2) - 2du(dv + 2\frac{v}{x}dx) + \frac{4}{3}k^{-1}xdy + 2x^4 du \tag{13}$$

where k is constant.

The null tetrad for the above line-element is

$$l^i = \frac{15}{4}kx\delta_2^i - \frac{257}{8}\sqrt{2}\delta_3^i - \frac{1}{\sqrt{2}x^2}\delta_4^i \,,\ n^i = -\sqrt{2}x^2\delta_3^i$$

$$m^i = (\frac{1}{2}kx - \frac{15}{4}i\sqrt{2})\delta_1^i - i(\frac{1}{2}kx + \frac{15}{4}\sqrt{2})\delta_2^i - (kv - \frac{59}{6}ix^2)\delta_3^i \qquad (14)$$

and the intrinsic derivatives are

$$D = \frac{15}{4}kx\frac{\partial}{\partial y} - \frac{257}{8}\sqrt{2}\frac{\partial}{\partial v} - \frac{1}{\sqrt{2}x^2}\frac{\partial}{\partial u} \,,\ \Delta = -\sqrt{2}x^2\frac{\partial}{\partial v}$$

$$\delta = (\frac{1}{2}kx - \frac{15}{4}i\sqrt{2})\frac{\partial}{\partial x} - i(\frac{1}{2}kx + \frac{15}{4}\sqrt{2})\frac{\partial}{\partial y} - (kv - \frac{59}{6}ix^2)\frac{\partial}{\partial v} \qquad (15)$$

so that the non-zero spin-coefficients are

$$\gamma = \frac{ik}{4}\sqrt{2} \,,\ \mu = \frac{ik}{2}49\sqrt{2} \,,\ \lambda = \frac{ik}{8}45\sqrt{2}$$

$$\frac{8}{153\nu} = \frac{\alpha}{2} = \tau = \beta = -\pi = \frac{k}{2} \qquad (16)$$

The non-zero component of Weyl scalar is

$$\Psi_3 = \frac{1}{2\sqrt{2}}ik^2 \qquad (17)$$

The Lanczos scalars for the metric (7), using (10) are found to be as

$$L_0 = L_1 = L_4 = 0$$

$$L_2 = -\frac{k}{6} \,,\ L_3 = \frac{ik}{8}45\sqrt{2} \,,\ L_5 = -\frac{k}{6} \qquad (18)$$

$$L_6 = -\frac{ik}{6}49\sqrt{2} \,,\ L_7 = \frac{153}{16}k$$

These equations clearly indicates that the Lanczos potential for Kaigorodov metric depends upon the constant k. If the constant k is chosen to be zero then $L_i = 0$ $(i = 1, 2, \ldots .7)$ showing thereby that the Kaigorodov spacetime is a flat spacetime.

(ii) A type III solution with twist

A type III solution with twist [15] is characterized by the following conditions:

$$\kappa = \sigma = \alpha = \beta = \gamma = \epsilon = \mu = \pi = 0$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0 \tag{19}$$

$$\rho , \lambda , \tau , \nu \neq 0 \; ; \; \Psi_3 \neq 0 \; ; \; \rho \neq \bar{\rho}$$

Debney, Wilkes and Zund [14] have exhibit a type III twist vacuum solution which is a manifestly different from Held [21] and Robinson [25]. Moreover, upon a particular specilization of the equations leading to this solution, Debney, Wilkes and Zund [15] also deduce the Held and Robinson solutions. The outline of this solution is given as follows:

Let $x^i = (u, r, x, y)$ denotes the real co-ordinate system, where r is radial parameter along the family of null congruence $\Gamma(l^i)$. The functions $U = \bar{U}$, $X^i = \bar{X}^i$, ω , ξ^i $(i = 1, 3, 4)$ are components of tetrad vctors and the covariant tetrad 1-forms

$$L \equiv l_i dx^i \; , \; N \equiv n_i dx^i \; , \; M \equiv m_i dx^i \; , \; \bar{M} \equiv \bar{m}_i dx^i \tag{20}$$

eventually leads to the metric

$$ds^2 = g_{ij} dx^i dx^j = LN + NL - M\bar{M} - \bar{M}M \tag{21}$$

The solution of NP field equations (c.f., [24]) leads to

$$\tau = \tau^\circ \rho \; , \; \lambda = \lambda^\circ \rho \; , \; \nu = \nu^\circ \rho$$

$$\omega = \omega^\circ \bar{\rho} \; , \; \xi^i = \xi^{oi} \bar{\rho} \; , \; \Psi_3 = \Psi_3^\circ \rho^2 \tag{22}$$

$$U = U^\circ + \omega^\circ \bar{\tau} + \bar{\omega}^\circ \tau \; , \; X^i = X^{oi} + \xi^{oi} \bar{\tau} + \bar{\xi}^{oi} \tau$$

where the 'degree sign' as a superscript indicates that function is independent of r and

$$\rho = \frac{-1}{r + i\Sigma} \quad \text{, where } \Sigma(u, x, y) \neq 0 \tag{23}$$

assures that the null ray congruence $\Gamma(l^i)$ having the tangent vector l^i has non-vanishing twist. Thus the covariant tetrad 1-form [cf., equation (20)] becomes

$$\begin{aligned} L &= (X^{\circ 1})^{-1}[du - \mathrm{Re}(\xi^{\circ 1})dx - \mathrm{Im}(\xi^{\circ 1})dy] \\ N &= dr - \mathrm{Re}(\omega^{\circ})dx - \mathrm{Im}(\omega^{\circ})dy - U^{\circ}L \\ M &= \frac{-1}{2\rho}[dx + idy] + \tau^{\circ}L \end{aligned} \tag{24}$$

with

$$\begin{aligned} \xi^{\circ 3} &= 1\,,\ \xi^{\circ 4} = i\,,\ X^{\circ 3} = X^{\circ 4} = 0\,,\ \xi^{\circ 1} = i\Omega \\ \lambda^{\circ} &= -\left(\frac{3}{4}\right)\frac{1}{x^2} = U^{\circ}\,,\ \tau^{\circ} = -\left(\frac{3}{2}\right)\frac{1}{x}\,,\ \Psi_3{}^{\circ} = -\left(\frac{3}{2}\right)\frac{1}{x^3} \\ \nu^{\circ} &= \lambda^{\circ}\tau^{\circ} + \Psi_3{}^{\circ}\,,\ X^{\circ 1} = x^{-\frac{3}{2}}\,,\ \omega^{\circ} = i(2\tau^{\circ}\Sigma - \eth\Sigma) \\ \eth &\equiv i\Omega\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \end{aligned} \tag{25}$$

where the real functions $\Omega(x, y, z)$ must be determined.
The non-zero spin coefficient, using equations (22) and (25), are found to be as

$$\lambda = -\left(\frac{3}{4}\right)\frac{\rho}{x^2}\,,\ \nu = -\left(\frac{3}{8}\right)\frac{\rho}{x^3}\,,\ \tau = -\left(\frac{3}{2}\right)\frac{\rho}{x} \tag{26}$$

The non-zero Weyl scalar is obtained as

$$\Psi_3 = -\left(\frac{3}{2}\right)\frac{\rho^2}{x^3} \tag{27}$$

Thus, the Lanczos scalars for the type III solution with twist are given by

$$\begin{aligned} L_0 &= L_2 = L_4 = L_6 = 0 \\ L_1 &= -\frac{1}{3}\rho\,,\ L_3 = -\frac{3}{4}\frac{\rho}{x^2} \\ L_5 &= \frac{1}{2x}\rho\,,\ L_7 = \frac{-3}{8}\frac{\rho}{x^3} \end{aligned} \tag{28}$$

which shows that the Lanczos potential depends only on one spin-coefficient ρ.

3. Conclusion

The present work exhibits the application of NP-formalism to determine the Lanczos scalars for several radiative spacetimes in general relativity. We note that these scalars are in terms of the corresponding spin-coefficients, and our conjecture is that it shall occur in any Petrov type if we select an adequate null tetrad.

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