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A Note on the Einstein spaces of Class Two

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Abstract
The necessary conditions for the embedding of algebraically special Einstein spacetimes of class two into E_6 are given.

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1. Introduction

It is known that [1] a V_4 is an Einstein spacetime if the Ricci and metric tensors are related through the equation

$$R_{ij} = \frac{R}{4}\, g_{ij} \tag{1}$$

One can easily prove that an Einstein space is a space of constant curvature (for the importance of spaces of constant curvature, see [1] and [2]).

Collinson [3] have obtained the necessary differential conditions that every class two algebraically special spacetime must satisfy. These conditions can be stated as the following

Lemma 1. If an empty spacetime V_4 is embedded into E_6 and
(i) is of Petrov type N or D, then $\kappa = \sigma = \rho - \bar{\rho} = 0$,
(ii) is of Petrov type II, then $\kappa = \sigma = \rho = 0$,
where κ, σ and ρ are associated to the Debever-Penrose vector.

On the other hand, Yakupov [5] and [6] proved the followings

Lemma 2. ([5]) Every Petrov type D or II Einstein spacetime of class two satisfies ${}^*C_2 = {}^*C_3 = 0$.

Lemma 3. ([6]) Any empty V_4 of Petrov type III can not be embedded into E_6.

Lemma 4. ([6]) In every empty spacetime which is embedded into E_6 and is of Petrov type II or D, the Ricci vector has zero curl.

Later on, Goenner [4] assured that Lemma 2 is true even without imposing the condition that V_4 satisfies equation (1). We do not agree with this result of Goenner and in this note we present a generalization to Lemmas 1 - 4. Since the calculations are very lengthy (although not so difficult) so we omit the details.

2. Einstein spaces embedded into E_6

The equations of Gauss, Codazzi and Ricci, which govern th embedding process have been studied using Newman-Penrose formalism. With this powerful tool, we have obtained the following theorems which are not found in the current litrature.

Theorem 1. In every Einstein spacetime V_4 of Petrov type II or D of class two, the Ricci vector is irrotational.

Theorem 2. In every algebraically special Einstein spacetime V_4, which is embedded into E_6, the Ricci vector does not rotate.

Theorem 3. If an Einstein spacetime V_4 is of class two and
(i) is of Petrov type D or N, then $\kappa = \sigma = \rho - \bar{\rho} = 0$,
(ii) is of Petrov type II or III, then $\kappa = \sigma = \rho = 0$.

Remarks.
(a) It may be noted that Theorem 1 includes Lemmas 2 and 4 as particular cases, while Theorem 3 generalizes Lemma 1.
(b) At present we are working with a more detailed analysis that may show the impossibility to embed every Einstein spacetime V_4 of Petrov type III into E_6. We are also seeking for more consistent arguments to prove that Lemma 2 can not be applied to any arbitrary V_4, according to the assertion of Goenner [5]. The details of these investigations will be published elsewhere.

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