Jour.Tensor Society **ISSN : 0974-5428**
Vol. 6 No. 2 (2012) 137-144
(S.I. Husain commemorative issue)

On Lightlike Submersions

Rachna Rani* and R. K. Nagaich**

*University College, Moonak, Punjab, India
E-mail: rachna_ucoe@yahoo.co.in
**Department of Mathematics, Punjabi University, Patiala, (Punjab), India
Email : nagaichrakesh@yahoo.com

Abstract
In this paper, we study lightlike submersions from a semi-Riemannian manifold onto a lightlike manifold having the dimension of radical distribution equal to one. Then we study O'Neill's tensors for such submersions and investigate their properties.

Keywords and phrases : Lightlike Manifold, Lightlike Submersion.
2000 AMS Subject Classification : 53C20, 53C50.

1. Introduction

The differential geometry of Riemannian immersions is known since the beginning of Riemannian geometry. But its dual notion of Riemannian submersion was first exposed in 1966 (Gray [5], O'Neill [6]). O'Neill [6] defined Riemannian Submersions as:

Let M and B be Riemannian manifolds. A Riemannian submersion $\pi : M \to B$ is a mapping of M onto B satisfying the following axioms $S.1$ and $S.2$:

S.1 π has maximal rank,

that is, each derivative map π_* of π is onto. Hence the implicit function theorem states that the fibre $\pi^{-1}(b)$ over any $b \in B$, is a closed submanifold of M of dimension = dim M - dim B. A vector field on M is called vertical if it is always tangent to the fibers and horizontal if orthogonal to the fibers.

S.2 π_* preserves the lengths of horizontal vectors.

A systematic exposition on Riemannian submersions can be found in Besses book [1]. Semi-Riemannian submersions were introduced by ONeill in [7] and are of interest in physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories. It is known that when M and

B are Riemannian manifolds, then the fibers are always Riemannian manifolds but when the manifolds are semi-Riemannian manifolds, then the fibers may not be semi-Riemannian. Recently, Sahin [8] defined a lightlike submersion from a semi-Riemannian manifold M to a lightlike manifold B.

2. Lightlike Manifolds

We recall notations and fundamental equations for lightlike manifolds from [2].

Let (M, g) be a real m-dimensional paracompact and smooth manifold where g is a symmetric tensor field of type (0, 2). Then, the radical or the null space of $T_x(M)$ is a subspace of $T_x(M)$, denoted by $RadT_x(M)$, and defined by

$$RadT_xM = \{\xi_x \in RadT_x(M); g(\xi_x, X) = 0, X \in T_xM\}. \tag{1}$$

The dimension, say r, of $RadT_x(M)$ is called nullity degree of g. If the mapping

$$RadTM : x \in M \longrightarrow RadT_xM, \tag{2}$$

defines a smooth distribution on M of rank $r > 0$ then $RadTM$ is called the radical distribution of rank r on M. Clearly, g is degenerate or non-degenerate on M if and only if $r > 0$ or $r = 0$, respectively. (M, g) is called a lightlike manifold if $0 < r \leq m$. Since M is paracompact therefore there exists a complementary distribution $S(TM)$ to $RadTM$ in TM and called screen distribution on M. Clearly, $S(TM)$ is semi-Riemannian therefore we have

$$TM = S(TM) \oplus RadTM. \tag{3}$$

The associated quadratic form h of type (p, q, r), where $p + q + r = m$, of g is a mapping $h : T_x(M) \to \Re$ given by $h(X) = g(X, X)$ for any $X \in T_x(M)$ and locally given by

$$h = -\sum_{a=1}^{q}(w^a)^2 + \sum_{A=q+1}^{q+p}(w^A)^2, \tag{4}$$

where $(w^1, ..., w^{p+q})$ are linearly independent local differential forms on M. With respect to local coordinate system (x^i), $i = 1, ..., m$, substitute $w^a = w_i^a dx^i$ and $w^A = w_i^A dx^i$ in (4), we get

$$h = g_{ij}dx^i dx^j, \quad \text{rank}|g_{ij}| = p + q < m, \tag{5}$$

$$g_{ij} = -\sum_{a=1}^{q} w_i^a w_j^a + \sum_{A=q+1}^{q+p} w_i^A w_j^A, \quad j \in \{1, ..., m\}. \tag{6}$$

Let the $r = 1$ then 1-dimensional radical distribution $RadTM$ is always integrable and we have the following theorem.

Theorem 2.1. [3] Let (M, g) be an m-dimensional lightlike manifold, with $RadTM$ of rank = 1. Then there exists a metric connection ∇ on M with respect to the degenerate metric tensor g.

3. Lightlike Submersions

For lightlike submersions, we follow [8]. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) an r-lightlike manifold. Consider a smooth submersion $f : M_1 \rightarrow M_2$, then $f^{-1}(p)$ is a submanifold of M_1 of dimension dinM_1 - dimM_2, for $p \in M_2$. The kernel of f_* at the point p is given by

$$Kerf_* = \{X \in T_p(M_1) : f_*(X) = 0\}, \tag{7}$$

and $(Kerf_*)^\perp$ is given by

$$(Kerf_*)^\perp = \{Y \in T_p(M_1) : g_1(Y, X) = 0, \forall X \in Kerf_*\}. \tag{8}$$

Since $T_p(M_1)$ is a semi-Riemannian vector space therefore $(Kerf_*)^\perp$ may not be a complement to $Kerf_*$ and assume $\triangle = Kerf_* \cap (Kerf_*)^\perp \neq \{0\}$. Thus we have the following four cases of submersions.

Case 1. When $0 < dim\triangle < min\{dim(Kerf_*), dim(Kerf_*)^\perp\}$

Then $\triangle$ is the radical subspace of $T_p(M_1)$. Since $Kerf_*$ is a real lightlike vector space, there is a complementary non-degenerate subspace to $\triangle$. Let $S(Kerf_*)$ be a complementary non-degenerate subspace to $\triangle$ in $Kerf_*$, therefore we have

$$Kerf_* = \triangle \perp S(Kerf_*). \tag{9}$$

Similarly

$$(Kerf_*)^\perp = \triangle \perp S(Kerf_*)^\perp, \tag{10}$$

where $S(Kerf_*)^\perp$ is a complementary subspace of $\triangle$ in $(Kerf_*)^\perp$. Since $S(Kerf_*)^\perp$ is non-degenerate in $T_p(M_1)$, therefore we have

$$T_p(M_1) = S(Kerf_*) \perp (S(Kerf_*))^\perp, \tag{11}$$

where $(S(Kerf_*))^\perp$ is the complementary subspace of $S(Kerf_*)$ in $T_p(M_1)$. Since $S(Kerf_*)$ and $(S(Kerf_*))^\perp$ are non-degenerate therefore we have

$$(S(Kerf_*))^\perp = S(Kerf_*)^\perp \perp (S(Kerf_*)^\perp)^\perp. \tag{12}$$

Then from [3], there exists a quasi-orthonormal basis of $T_p(M_1)$ along $Kerf_*$, we have

$$g(\xi_i, \xi_j) = g(N_i, N_j) = 0; \quad g(\xi_i, N_j) = \delta_{ij}, \tag{13}$$

$$g(W_\alpha, \xi_j) = g(W_\alpha, N_j) = 0; \quad g(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta}, \tag{14}$$

for any $i, j \in \{1, ..., r\}$ and $\alpha, \beta \in \{1, ..., t\}$, where $\{N_i\}$ are smooth lightlike vector fields of $(S(Kerf_*)^{\perp})^{\perp}$, $\{\xi_i\}$ is basis of $\triangle$ and W_α is a basis of $S(Kerf_*)^{\perp}$. Denote the set of vector fields $\{N_i\}$ by $ltr(Kerf_*)$ and consider

$$tr(Kerf_*) = ltr(Kerf_*) \perp S(Kerf_*)^{\perp}. \quad (15)$$

Using (13), it is clear that $ltr(Kerf_*)$ and $Ker(f_*)$ are not orthogonal to each other. Denote $\mathcal{V} = Kerf_*$, the vertical space of $T_p(M_1)$ and $\mathcal{H} = tr(Kerf_*)$, the horizontal space then we have

$$T_p(M_1) = \mathcal{V}_p \oplus \mathcal{H}_p. \quad (16)$$

Definition 3.1. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) an r-lightlike manifold. Let $f : M_1 \to M_2$ be a submersion such that

(a) $dim\triangle = dim\{(Kerf_*) \cap (Kerf_*)^{\perp}\} = r$,
$0 < r < min\{dim(Kerf_*), dim(Kerf_*)^{\perp}\}$.

(b) f_* preserves the length of horizontal vectors, that is, $g_1(X, Y) = g_2(f_*X, f_*Y)$ for $X, Y \in \Gamma(\mathcal{H})$.

Then f is called an r-lightlike submersion.
Case 2. When $dim\triangle = dim(Kerf_*) < dim(Kerf_*)^{\perp}$.
Then $\mathcal{V} = \triangle$ and $\mathcal{H} = S(Kerf_*)^{\perp} \perp ltr(Kerf_*)$ and f is called an isotropic submersion.
Case 3. When $dim\triangle = dim(Kerf_*)^{\perp} < dim(Kerf_*)$.
Then $\mathcal{V} = S(Kerf_*) \perp \triangle$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a co-isotropic submersion.
Case 4. When $dim\triangle = dim(Kerf_*) = dim(Kerf_*)^{\perp}$.
Then $\mathcal{V} = \triangle$ and $\mathcal{H} = ltr(Kerf_*)$ and f is called a totally lightlike submersion.

A basic vector field on M_1 is a horizontal vector field X which is f-related to a vector field $\tilde{X}$ on M_2, that is, $f_*(X_p) = \tilde{X}_{f(p)}$ for all $p \in M_1$. Every vector field $\tilde{X}$ on M_2 has a unique horizontal lift X to M_1 and X is basic. Therefore $X \leftrightarrow \tilde{X}$ is a one to one correspondence between basic vector fields on M_1 and arbitrary vector field on M_2.
Example. Let $\Re^4_{0,1,3}$ and $\Re^2_{0,1,0}$ be $\Re^4$ and $\Re^2$ endowed with the Lorentzian metric $g_1 = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2$, and degenerate metric $g_2 = (dy_2)^2$, where x_1, x_2, x_3, x_4 and y_1, y_2 are the canonical coordinates on $\Re^4$ and $\Re^2$, respectively. Define a map

$$f : \Re^4_{0,1,3} \to \Re^2_{0,1,0}, \quad (x_1, x_2, x_3, x_4) \to (x_1 + x_3, \frac{x_2 + x_4}{\sqrt{2}}).$$

Then the kernel of f_* is given by

$$Kerf_* = Span\{W_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad W_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\},$$

and

$$(Kerf_*)^{\perp} = Span\{T_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, \quad T_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\}.$$

Clearly, we have $W_1 = T_1$, therefore

$$\Delta = kerf_* \cap (Kerf_*)^{\perp} = Span\{W_1\}.$$

Then $lrt(Kerf_*) = Span\{N = \frac{1}{2}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3})\}$. Easily we can show that $g_1(N, W_1) = 1$ and $g_1(N, W_2) = 0$. Thus the horizontal and vertical spaces are given by

$$\mathcal{V} = Span\{W_1, W_2\}, \quad \mathcal{H} = Span\{T_2, N\}.$$

Also $f_*(T_2) = \sqrt{2}\frac{\partial}{\partial y_2}$, $f_*(N) = \frac{\partial}{\partial y_1}$. We also obtain that

$$g_1(N, N) = g_2(f_*N, f_*N) = 0.$$

$$g_1(T_2, T_2) = g_2(f_*T_2, f_*T_2) = 0.$$

Hence, f is a 1-lightlike submersion.

Let $h : TM_1 \to \mathcal{H}$ and $\nu : TM_1 \to \mathcal{V}$ denote the projections associated with the direct sum decomposition $TM_1 = \mathcal{H} \oplus \mathcal{V}$.

Theorem 3.2. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) be a 1-lightlike manifold. Let $f : M_1 \to M_2$ be a lightlike submersion and denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M_1 and M_2, respectively. If X, Y are vector fields, f-related to $\tilde{X}, \tilde{Y}$ then

(i) $g_1(X, Y) = g_2(\tilde{X}, \tilde{Y}) o f$.

(ii) $h[X, Y]$ is the basic vector field, f-related to $[\tilde{X}, \tilde{Y}]$.

(iii) $h(\nabla_X Y)$ is the basic vector field, f-related to $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$.

(iv) For any vertical vector field V, $[X, V]$ is vertical.

Proof. Property (i) immediately follows from the (b) of the definition 3.1. Property (ii) follows from $f_*[X, Y] = [\tilde{X}, \tilde{Y}]$. Now from the Kozsul formula, we have

$$\begin{aligned} 2g_1(\nabla_X Y, Z) &= X(g_1(Y, Z)) + Y(g_1(Z, X)) - Z(g_1(X, Y)) - g_1(X, [Y, Z]) \\ &\quad + g_1([Z, X], Y) + g_1(Z, [X, Y]), \end{aligned} \tag{17}$$

for any $X, Y, Z \in \Gamma(TM_1)$. Considering X, Y, Z as the horizontal lifts of the vector fields $\tilde{X}, \tilde{Y}, \tilde{Z}$, respectively then we have $X(g_1(Y, Z)) = \tilde{X}(g_2(\tilde{Y}, \tilde{Z})) o f$ and $g_1([X, Y], Z) = g_2([\tilde{X}, \tilde{Y}], \tilde{Z}) o f$. Then (17) becomes

$$\begin{aligned} 2g_1(\nabla_X Y, Z) &= \tilde{X}(g_2(\tilde{Y}, \tilde{Z})) o f + \tilde{Y}(g_2(\tilde{Z}, \tilde{X})) o f - \tilde{Z}(g_2(\tilde{X}, \tilde{Y})) o f \\ &\quad - g_2(\tilde{X}, [\tilde{Y}, \tilde{Z}]) o f + g_2([\tilde{Z}, \tilde{X}], \tilde{Y}) o f + g_2(\tilde{Z}, [\tilde{X}, \tilde{Y}]) o f \\ &= 2g_2(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) o f. \end{aligned} \tag{18}$$

Thus from (18), the property (iii) follows, since f is surjective and $\tilde{Z}$ is arbitrarily chosen. Now, let $V \in \Gamma(\mathcal{V})$ then $[X, V]$ is f-related to $[\tilde{X}, 0]$, hence (iv) follows.

4. Fundamental Tensors or Invariants for Lightlike Submersions

In this section, we define O'Neill's [6] tensor for lightlike submersions. Let ∇ be the Levi-Civita connection of (M_1, g_1), then define a tensor field T of type $(1, 2)$ by

$$T_X Y = h\nabla_{\nu X}\nu Y + \nu\nabla_{\nu X}hY. \tag{19}$$

It is easy to prove that T satisfies the following properties.

(i) T is a vertical tensor field, that is, $T_X = T_{\nu X}, \forall X, Y \in \Gamma(TM_1)$.

(ii) T reverses the horizontal and vertical subspaces, that is, $T_x(\mathcal{V}_p) \subseteq \mathcal{H}_p$, $T_x(\mathcal{H}_p) \subseteq \mathcal{V}_p$, $x \in T_p(M_1)$.

(iii) The integrability of the vertical distribution implies that T has symmetry property for vertical vector fields, that is, $T_V W = T_W V, \forall V, W \in \Gamma(\mathcal{V})$.

The other tensor is defined as

$$A_X Y = h\nabla_{hX}\nu Y + \nu\nabla_{hX}hY. \tag{20}$$

Again A is a $(1, 2)$-tensor and has following properties

(i) A is a horizontal tensor field, that is, $A_X = A_{hX}, \forall X, Y \in \Gamma(TM_1)$.

(ii) A also reverses the horizontal and vertical subspaces.

It should be noted that the tensor fields T and A are skew-symmetric in the Riemannian submersions but not in the case of lightlike submersions because the horizontal and vertical subspaces are not orthogonal. In fact we have

Theorem 4.1. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) be a 1-lightlike manifold. Let $f : M_1 \to M_2$ be a lightlike submersion then

(i) $g_1(T_V X, Y) + g_1(X, T_V Y) = 0$,

(ii) $g_1(A_Z X, Y) + g_1(X, A_Z Y) = 0$,

for any $V \in \Gamma(Kerf_*)$, $Z \in \Gamma(tr(Kerf_*))$ and $X, Y \in \Gamma(ltr(Kerf_*))$ or $X \in \Gamma(S(Kerf_*))$ and $Y \in \Gamma(S(Kerf_*)^{\perp})$ and vice-versa.

Proof. We prove (i) in two different cases.

Case 1. Let $X, Y \in \Gamma(ltr(Kerf_*))$ and $V \in \Gamma(Kerf_*)$, then using (19) we get

$$T_V X = h\nabla_V \nu X + \nu\nabla_V hX = \nu\nabla_V X, \tag{21}$$

and

$$T_V Y = h\nabla_V \nu Y + \nu\nabla_V hY = \nu\nabla_V Y. \quad (22)$$

Since $\nabla g_1 = 0$ therefore we get

$$V g_1(X, Y) = g_1(\nabla_V X, Y) + g_1(X, \nabla_V Y), \quad (23)$$

then using (21)-(23), we obtain the result.
Case 2. Let $X \in \Gamma(S(Ker f_*))$ and $Y \in \Gamma(S(Ker f_*)^\perp)$ and $V \in \Gamma(Ker f_*)$, then using (19) we get

$$T_V X = h\nabla_V \nu X + \nu\nabla_V hX = h\nabla_V X. \quad (24)$$

and

$$T_V Y = h\nabla_V \nu Y + \nu\nabla_V hY = \nu\nabla_V Y, \quad (25)$$

then using (23)-(25), the result follows. Similarly, we can prove (i) when $X \in \Gamma(S(Ker f_*)^\perp)$ and $Y \in \Gamma(S(Ker f_*))$

The proof of (ii) is similar to that of (i).
Theorem 4.2. Let (M_1, g_1) be a semi-Riemannian manifold and (M_2, g_2) be a 1-lightlike manifold. Let $f : M_1 \to M_2$ be a lightlike submersion then

(i) $g_1(T_U V, X) + g_1(V, T_U X) = 0,$

(ii) $g_1(A_X Y, V) + g_1(Y, A_X V) = 0,$

for any $X, Y \in \Gamma(S(Ker f_*)^\perp)$ and $U, V \in \Gamma(S(Ker f_*))$.
Proof. We only prove (i), the proof of (ii) being similar. Using (19) we get

$$T_U V = h\nabla_U \nu V + \nu\nabla_U hV = h\nabla_U V. \quad (26)$$

and

$$T_U X = h\nabla_U \nu X + \nu\nabla_U hX = \nu\nabla_U X. \quad (27)$$

Since $\nabla g_1 = 0$ therefore we get

$$U g_1(V, X) = g_1(\nabla_U V, X) + g_1(V, \nabla_U X), \quad (28)$$

then using (26)-(28), we obtain the result.

Form the above theorem, we may obtain $T_U X$ from g_1 and $T_U V$ and $A_X V$ from g_1 and $A_X Y$, where $X, Y \in \Gamma(S(Ker f_*)^\perp)$.

From (19) and (20), we have the following.
Lemma 4.3. Let $f : M_1 \to M_2$ be a lightlike submersion then

(i) $\nabla_U V = T_U V + \nu\nabla_U V,$

(ii) $\nabla_V X = h\nabla_V X + T_V X$,

(iii) $\nabla_X V = A_X V + \nu\nabla_X V$,

(iv) $\nabla_X Y = h\nabla_X Y + A_X Y$,

for any $X, Y \in \Gamma(tr(Kerf_*))$ and $U, V \in \Gamma(Kerf_*)$.

References

1. Besse, A. L. *Einstein manifolds*, Springer-Verlag, Berlin, 1987.
2. Duggal, K. L. *Warped Product of Lightlike Manifolds*, Nonlinear Analysis, **47** (2001), 3061–3072.
3. Duggal, K. L. and Bejancu, A. *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Vol. 364 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.
4. Falcitelli, M., Ianus, S. and Pastore, A. M. *Riemannian Submersions and Related Topics*, World Scientific, 2004.
5. Gray, A. *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech., **16** (1967), 715–737.
6. O'Neill, B. *The fundamental equations of a submersion*, Michigan Math. J., **13** (1966), 459–469.
7. O'Neill, B. *Semi-Riemannian Geometry with Applictions to Relativity*, New York, Academic Press (1983).
8. Sahin, B. and Gunduzalp, Y. *Submersion from semi-Riemannian manifolds onto lightlike manifolds*, Hacettepe Journal of Mathematics and Statistics, **39(1)** (2010), 41–53.