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Totally Umbilical Lightlike Hypersurfaces of Semi-Riemannian Manifolds

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Abstract

This paper deals with totally umbilical lightlike hypersurfaces of semi-Riemannian manifolds. We obtain theorems on parallelism of local second fundamental form, geodesibility of lightlike hypersurfaces and screen distribution. We prove a necessary and sufficient condition for the integrability of screen distribution. We also study lightlike hypersurfaces with totally umbilical screen distribution in indefinite real space form.

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1. Introduction

The geometry of lightlike hypersurfaces is used in general relativity as they produce models of different type of horizons (event horizons, Cauchy's horizons, Kruskal's horizons). This geometry is also applicable to study a class of lightlike hypersurfaces of 4-dimensional electromagnetic spacetimes. In [1] Duggal and Bejancu showed that a class of electromagnetic invariant lightlike hypersurfaces comes from the physically significant homogeneous spacetime manifold of general relativity. But the geometry of lightlike hypersurfaces is different from classical theory of hypersurfaces and rather difficult. The difference occur mainly due to the fact that, in case of lightlike hypersurfaces, the normal bundle and the tangent bundle have non-zero intersection. Moreover, the normal bundle is same as the null tangent bundle along a non-zero distribution, called radical distribution. In this paper after brief resume of basic theory and notations, we syudy parallelism of local second fundamental form, geodesibility of lightlike hypersurfaces. In §4, we discuss totally umbilical screen distribution of lightlike hypersurfaces of semi-Riemannian manifolds. We obtain Lie derivatives of local second fundamental form and induced metric g and use to find a condition for the geodesibility of screen distribution. Finally, we obtain a necessary and sufficient condition for the geodesibility of lightlike hypersurfaces with totally umbilical screen distribution in indefinite real space form.

2. Preliminaries

We recall notations and fundamental equations for lightlike hypersurfaces, which are due to the book [1] by Duggal and Bejancu.

Let M be a hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ of index $q \in \{1,2,...,m+1\}$. The hypersurface is said to be lightlike hypersurface if the normal vector bundle $TM^{\perp}$ is same as the null tangent bundle along a non-zero distribution, called a radical distribution $Rad(TM)$ of M, where

$$Rad(T_uM) = \{V_u \in T_u\bar{M} : \bar{g}(V_u, W_u) = 0, \forall \quad W_u \in T_uM\}.$$

There exists a Riemannian screen distribution $S(TM)$ which is complementary to the radical distribution in TM therefore

$$TM = S(TM) \perp TM^{\perp}. \tag{1}$$

From Theorem 1.1 in [1] at page 79, it is clear that for a screen distribution $S(TM)$ on M there exists a unique vector bundle $tr(TM)$ known as lightlike transversal vector bundle such that for any non-zero local normal section ξ of $Rad(TM)$ there exists a unique section N of $tr(TM)$, we have

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, W) = 0, \forall \quad W \in \Gamma(S(TM)). \tag{2}$$

Then we have the decomposition

$$T\bar{M}|_M = S(TM)\perp(TM^\perp \oplus tr(TM)) = TM \oplus tr(TM) \tag{3}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then using the decomposition in (3), we obtain Gauss and Weingarten formulae as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{5}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while $h(X, Y)$ and $\nabla^t_X V$ belong to $\Gamma(tr(TM))$. Here ∇ is a torsion free linear connection on M, h is a $\Gamma(tr(TM))$-valued symmetric bilinear form on $\Gamma(TM)$ and known as the second fundamental form. A_V is a linear operator on $\Gamma(TM)$ and known as the shape operator of lightlike immersion and ∇^t is a linear connection on $\Gamma(tr(TM))$.

Locally for the pair $\{\xi, N\}$, we define local second fundamental form B and 1-form τ as

$$B(X, Y) = \bar{g}(h(X, Y), \xi) \quad \text{therefore} \quad h(X, Y) = B(X, Y)N. \tag{6}$$

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi) \quad \text{therefore} \quad \nabla^t_X N = \tau(X)N. \tag{7}$$

Hence locally, (4) and (5) become

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{8}$$

$$\bar{\nabla}_X V = -A_V X + \tau(X)N. \tag{9}$$

If P denotes the projection morphism of TM on $S(TM)$ then from (1) we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \tag{10}$$

$$\nabla_X U = -A^*_U X + \nabla^{*t}_X U, \tag{11}$$

for any $X, Y \in \Gamma(TM)$, $U \in \Gamma(TM^\perp)$. Where $\nabla^*_X PY$ and $A^*_U X$ belong to $\Gamma(S(TM))$ while $h^*(X, PY)$ and $\nabla^{*t}_X U$ belong to $\Gamma(TM^\perp)$. Here h^* and A^*_U are called the second fundamental form and the shape operator of screen distribution, respectively. By direct calculation, we have

$$g(A_V Y, PW) = g(V, h^*(Y, PW)), \quad g(A_V Y, V) = 0, \tag{12}$$

$$g(A^*_U X, PY) = g(U, h(X, PY)), \quad g(A^*_U X, V) = 0, \tag{13}$$

for any $X, Y \in \Gamma(TM)$, $U \in \Gamma(TM^\perp)$ and $V \in \Gamma(tr(TM))$. Locally define

$$C(X, PY) = \bar{g}(h^*(X, PY), N) \quad \text{therefore} \quad h^*(X, PY) = C(X, PY)\xi. \tag{14}$$

$$\epsilon(X) = \bar{g}(\nabla^{*t}_X \xi, N) \quad \text{therefore} \quad \nabla^{*t}_X \xi = \epsilon(X)\xi, \tag{15}$$

then (10) and (11) become

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{16}$$

$$\nabla_X \xi = -A^*_\xi X + \epsilon(X)\xi = -A^*_\xi X - \tau(X)\xi. \tag{17}$$

and also have

$$g(A_N Y, PW) = C(Y, PW)), \quad \bar{g}(A_N Y, N) = 0, \tag{18}$$

$$g(A^*_\xi X, PY) = B(X, PY), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{19}$$

here (19) implies that

$$B(X, A^*_\xi Y) = B(Y, A^*_\xi X). \tag{20}$$

From (6) it is clear that

$$B(X, \xi) = 0, \tag{21}$$

that is, the second fundamental form of a lightlike hypersurface is degenerate.
Let define 1-form η as

$$\eta(X) = \bar{g}(X, N), \tag{22}$$

then, we have

Theorem 2.1. [1]: (i) The linear connection ∇^* is a metric connection on $S(TM)$.
(ii) The induced connection ∇ on M, satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{23}$$

for any $X, Y \in \Gamma(TM)$.
From (22), it is clear that $X \in S(TM)$ if and only if $\eta(X) = 0$.
Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections on $\bar{M}$ and the induced connection on M, respectively. Denote $\bar{R}$ and R the curvature tensors of $\bar{\nabla}$ and ∇, respectively then by using (4) and (5), we have

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + A_{h(X,Z)}Y - A_{(Y,Z)}X - (\nabla_Y h)(X, Z) \\ &\quad + (\nabla_X h)(Y, Z), \end{aligned} \tag{24}$$

for any $X, Y, Z \in \Gamma(TM)$, where

$$(\nabla_X h)(Y, Z) = \nabla^t_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{25}$$

Using (6), (7) and (14), we obtain

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z, PW) &= g(R(X,Y)Z, PW) + B(X,Z)C(Y,PW) \\ &\quad -B(Y,Z)C(X,PW),\end{aligned} \tag{26}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)Z, \xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) \\ &\quad -B(X,Z)\tau(Y),\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{g}(\bar{R}(X,Y)PZ, N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - C(Y,PZ)\tau(X) \\ &\quad +C(X,PZ)\tau(Y),\end{aligned} \tag{28}$$

and

$$\bar{g}(\bar{R}(X,Y)\xi, N) = C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X,Y), \tag{29}$$

where

$$(\nabla_X B)(Y,Z) = X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \tag{30}$$

$$(\nabla_X C)(Y,PZ) = X(C(Y,PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ). \tag{31}$$

3. Totally Umbilical Lightlike Hypersurface

Let $(M, g, S(TM))$ be a lightlike hypersurface of a $(m+2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ then M is said to be totally umbilical if and only if there exists a smooth function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall \quad X, Y \in \Gamma(TM). \tag{32}$$

M is said to be totally geodesic if $\rho = 0$.
The local second fundamental form B is said to be parallel with respect to ∇ if

$$(\nabla_X B)(Y,Z) = 0, \quad \forall \quad X, Y, Z \in \Gamma(TM). \tag{33}$$

Let M be a totally umbilical lightlike hypersurface of $\bar{M}$ then using (30) and (32) we have $(\nabla_X B)(Y,Z) = \rho\{(\nabla_X g)(Y,Z)\} = \rho\{B(X,Y)\eta(Z)+$

$B(X,Z)\eta(Y)\}$, by virtue of (23). Replace Z by ξ and using (21), we have $(\nabla_X B)(Y,\xi) = \rho B(X,Y)$, therefore we have the following

Theorem 3.1. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then M is a totally geodesic if and only if the local second fundamental form B is parallel.

Theorem 3.2. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the shape operator of the screen distribution is symmetric with respect to ∇, if the local second fundamental form B is parallel.

Proof. Let B is parallel with respect to ∇ therefore $(\nabla_X B)(\xi, Y) = 0$ then $B(X,\xi) = 0$ implies that

$$B(\nabla_X\xi, Y) + B(\xi, \nabla_X Y) = 0. \tag{34}$$

Similarly $(\nabla_Y B)(\xi, X) = 0$ implies that

$$B(\nabla_Y\xi, X) + B(\xi, \nabla_Y X) = 0. \tag{35}$$

Since $B(X,\xi) = 0$ therefore from (34) and (35), we obtain $B(\nabla_X\xi, Y) = B(\nabla_Y\xi, X)$. Using the hypothesis that M is a totally umbilical lightlike hypersurface therefore $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X)$ then using (17), we have $g(A^*_\xi X, Y) + \tau(X)g(\xi, Y) = g(A^*_Y\xi, X) + \tau(Y)g(\xi, X)$. Let $X, Y \in \Gamma(S(TM))$ then we get $g(A^*_\xi X, Y) = g(A^*_Y\xi, X)$, which completes the proof.

Next, since $\bar{\nabla}$ is a metric connection therefore for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$ we have $(\bar{\nabla}_X g)(Y,\xi) = 0$ then using (2) and (21), we obtain $Xg(Y,\xi) - g(\nabla_X Y, \xi) - B(X,Y) - g(Y, \nabla_X\xi) = 0$. Let $Y \in \Gamma(S(TM))$ then using (16) and (17), we get

$$B(X,Y) = g(Y, A^*_\xi X). \tag{36}$$

Let M be a totally umbilical lightlike hypersurface of $\bar{M}$ then (36) gives $g(Y, \rho X - A^*_\xi X) = 0$ then non degeneracy of $S(TM)$ implies that

$$A^*_\xi X = \rho X, \tag{37}$$

therefore from (36) and (37), we have the following

Theorem 3.3. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then M is totally umbilical if and only if $A^*_\xi X = \rho X$.

Theorem 3.4. Let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$. Then the function ρ satisfies the following partial differential equation

$$\xi(\rho) - \rho^2 + \rho\tau(\xi) = 0. \tag{38}$$

Proof. Since $\bar{M}$ is of constant curvature $\bar{c}$ therefore

$$\bar{R}(X,Y)Z = \bar{c}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\}, \tag{39}$$

then using (39) in (27), we have

$$\begin{aligned} &g(Y,Z)\{\bar{c}g(\xi,X) - X(\rho) - \rho\tau(X)\} - g(X,Z)\{\bar{c}g(\xi,Y) - Y(\rho) - \rho\tau(Y)\} \\ &= \rho\{g(\nabla_Y X, Z) + g(X, \nabla_Y Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)\}. \end{aligned} \tag{40}$$

Replace Z by PZ in above equation, we get

$$\begin{aligned} &g(Y,PZ)\{\bar{c}g(\xi,X) - X(\rho) - \rho\tau(X)\} - g(X,PZ)\{\bar{c}g(\xi,Y) \\ &-Y(\rho) - \rho\tau(Y)\} = \rho\{g(\nabla_Y X, PZ) + g(X, \nabla_Y PZ) \\ &-g(\nabla_X Y, PZ) - g(Y, \nabla_X PZ)\}. \end{aligned} \tag{41}$$

Now $g(\nabla_X Y, PZ) = -g(Y, \bar{\nabla}_X PZ) = -g(Y, \nabla_X PZ) - B(X, PZ)\eta(Y)$, this implies

$$g(\nabla_X Y, PZ) + g(Y, \nabla_X PZ) = -\rho g(X, PZ)\eta(Y), \tag{42}$$

similarly

$$g(\nabla_Y X, PZ) + g(X, \nabla_Y PZ) = -\rho g(Y, PZ)\eta(X). \tag{43}$$

Using (42) and (43) in (41), we have

$$\begin{aligned} &g(Y,PZ)\{\bar{c}g(\xi,X) - X(\rho) - \rho\tau(X) + \rho^2\eta(X)\} - g(X,PZ)\{\bar{c}g(\xi,Y) \\ &-Y(\rho) - \rho\tau(Y) + \rho^2\eta(Y)\} = 0. \end{aligned} \tag{44}$$

Let $X = \xi$ and $Y \in \Gamma(S(TM))$ we obtain

$$g(PY, PZ)\{-\xi(\rho) - \rho\tau(\xi) + \rho^2\} = 0, \tag{45}$$

then non degeneracy of $\Gamma(S(TM))$ gives the proof.

In particular, if we assume $X = PX$ and $Y = PY$ in (44) then we have

$$PX(\rho) + \rho\tau(PX) = 0. \tag{46}$$

Next, let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold of constant curvature $\bar{c}$ then using (18) in (29), we obtain

$$\begin{aligned} \bar{c}\{g(Y,\xi)\eta(X) - g(X,\xi)\eta(Y)\} &= g(A_N Y, A^*_\xi X) - g(A_N X, A^*_\xi Y) \\ &\quad -2d\tau(X,Y). \end{aligned} \tag{47}$$

By virtue of Theorem 3.3 in above equation, we obtain

$$\bar{c}\{g(Y,\xi)\eta(X) - g(X,\xi)\eta(Y)\} = \rho g(A_N Y, X) - \rho g(A_N X, Y) - 2d\tau(X,Y). \quad (48)$$

Let $X, Y \in \Gamma(S(TM))$, we have

$$2d\tau(X,Y) = \rho\{g(A_N Y, X) - g(A_N X, Y)\}, \quad (49)$$

then from Theorem 2.3 at page 89 of [1], we have the following

Theorem 3.5. let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$. Then $S(TM)$ is integrable if and only if each 1-form τ is induced by $S(TM)$ is closed, that is, $d\tau = 0$.

Theorem 3.6. **([1]):** let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the Ricci tensor of the induced connection ∇ is symmetric if and only if each 1-form τ induced by $S(TM)$ is closed.

Therefore from Theorems 3.5 and 3.6, we have the following

Theorem 3.7. Let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$. Then following assertions are equivalent:

(i) $S(TM)$ is an integrable distribution.

(ii) each 1-form τ induced by $S(TM)$ is closed.

(iii) the Ricci tensor of induced connection ∇ is symmetric.

4. Totally Umbilical Screen Distribution

The screen distribution $S(TM)$ is said to be totally umbilical if on any coordinate neighborhood $u \subset M$, there exists a smooth function λ such that

$$C(X, PY) = \lambda g(X, PY), \quad \forall \quad X, Y \in \Gamma(TM|_u). \quad (50)$$

Clearly C is symmetric on $S(TM|_u)$ and hence $S(TM)$ is integrable, according to Theorem 2.3 at page 89 of [1]. If $\lambda = 0$ then $S(TM)$ is said to be totally geodesic. Also using (18) and (50), we have

$$A_N X = \lambda PX, \quad \text{and} \quad C(\xi, PX) = 0, \quad (51)$$

for any $X \in \Gamma(TM|_u)$.

Using (17) and (19) we have $(L_\xi B)(X,Y) = (\nabla_\xi B)(X,Y) - B(A^*_\xi X, Y) - B(X, A^*_\xi Y)$. Since $B(X, PY) = g(A^*_\xi X, PY)$, then using (19), we have

$$(L_\xi B)(X,Y) = (\nabla_\xi B)(X,Y) - 2B(A^*_\xi X, Y), \quad (52)$$

thus we have the following

Lemma 4.1. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the local second fundamental form B is parallel then

$$(L_\xi B)(X, Y) = -2B(A^*_\xi X, Y). \tag{53}$$

Theorem 4.2. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the Lie derivative of local second fundamental form B with respect to ξ is given by

$$(L_\xi B)(X, Y) = -\tau(\xi)B(X, Y) - B(A^*_\xi X, Y). \tag{54}$$

Proof. Let $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$ then using (17) and (19), we have

$$(\nabla_X B)(\xi, Y) = B(A^*_\xi X, Y), \tag{55}$$

and using (52), we have

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) + 2B(A^*_\xi X, Y), \tag{56}$$

therefore, from (55) and (56), we have

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = (L_\xi B)(X, Y) + B(A^*_\xi X, Y). \tag{57}$$

By direct calculations, using (8), (9) and (30), we obtain

$$\begin{aligned} \bar{R}(X, Y)Z \;=\; & R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X + \{(\nabla_X B)(Y, Z) \\ & -(\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\}N. \end{aligned} \tag{58}$$

Taking transversal components, we get

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z), \tag{59}$$

replacing X by ξ, Y by X and Z by Y, we obtain

$$\begin{aligned} (\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) &= \tau(X)B(\xi, Y) - \tau(\xi)B(X, Y) \\ &= -\tau(\xi)B(X, Y), \end{aligned} \tag{60}$$

therefore from (57) and (60), the theorem follows

Lemma 4.3. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then Lie derivative of g with respect to a vector filed $N \in \Gamma(tr(TM))$ is given by

$$(L_N g)(X,Y) = X\eta(Y) + Y\eta(X) + \eta([X,Y]) - 2\eta(\nabla_X Y). \qquad (61)$$

Proof. Let $X, Y \in \Gamma(TM)$ then

$$\begin{aligned} \eta(\nabla_X Y) &= g(\nabla_X Y, N) = g(\bar{\nabla}_X Y, N) \\ &= Xg(Y,N) - g(Y, \bar{\nabla}_X N) \\ &= X\eta(Y) - g(Y,[X,N]) - g(Y, \bar{\nabla}_N X) \\ &= X\eta(Y) + g(Y,[N,X]) - Ng(Y,X) + g(X, \bar{\nabla}_N Y) \\ &= X\eta(Y) + g(Y,[N,X]) - Ng(Y,X) + g(X,[N,Y]) \\ &= +g(X, \bar{\nabla}_Y N) X\eta(Y) - (L_N g)(X,Y) + g(X, \bar{\nabla}_Y N) \\ &= X\eta(Y) - (L_N g)(X,Y) + Yg(X,N) - g(\bar{\nabla}_Y X, N) \\ &= X\eta(Y) + Y\eta(X) - (L_N g)(X,Y) - \eta(\nabla_X Y) + \eta([X,Y]), \end{aligned}$$

this gives the proof.

Lemma 4.4. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. If the screen distribution $S(TM)$ is totally geodesic then $N \in \Gamma(tr(TM))$ is $S(TM)$-Killing vector field.

Proof. Let $S(TM)$ be a totally geodesic then for any $X, Y \in \Gamma(S(TM))$, $C(X,Y) = 0$ implies that

$$\eta(\nabla_X Y) = g(\nabla^*_X Y + C(X,Y)\xi, N) = C(X,Y) = 0, \qquad (62)$$

therefore from (61) we have $(L_N g)(X,Y) = \eta([X,Y])$ this implies $(L_N g)(X,Y) = -(L_N g)(Y,X)$. But on the other hand, using (62), we have $(L_N g)(X,Y) - (L_N g)(Y,X) = 2\eta([X,Y]) - 2\eta(\nabla_X Y) - 2\eta(\nabla_Y X) = 0$ therefore $(L_N g)(X,Y) = (L_N g)(Y,X) = -(L_N g)(X,Y)$. Thus $(L_N g)(X,Y) = 0$, this completes the proof.

Theorem 4.5. Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$ of constant curvature $\bar{c}$ and screen distribution be totally umbilical. Then M is totally geodesic immersed in $\bar{M}(\bar{c})$ if and only if λ is a solution of the partial differential equation

$$\xi(\lambda) - \lambda\tau(\xi) - (c + \lambda^2) = 0. \qquad (63)$$

Proof. Using the hypothesis that screen distribution is totally umbilical with (31), we have

$$\begin{aligned}
&(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ) \\
&= g(Y,PZ)X(\lambda) - g(\nabla_X Y,PZ)\lambda - g(Y,\nabla_X PZ)\lambda + C(X,PZ)\eta(Y)\lambda \\
&\quad -g(X,PZ)Y(\lambda) + g(\nabla_Y X,PZ)\lambda + g(X,\nabla_Y PZ)\lambda - C(Y,PZ)\eta(X)\lambda \\
&\quad +\tau(Y)C(X,PZ) - \tau(X)C(Y,PZ) \\
&= g(Y,Z)X(\lambda) - g(\nabla_X Y,PZ)\lambda + g(\bar{\nabla}_X Y,PZ)\lambda + B(X,PZ)\eta(Y)\lambda \\
&\quad +C(X,PZ)\eta(Y)\lambda - g(X,PZ)Y(\lambda) + g(\nabla_Y X,PZ)\lambda - g(\bar{\nabla}_Y X,PZ)\lambda \\
&\quad -B(Y,PZ)\eta(X)\lambda - C(Y,PZ)\eta(X)\lambda + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ) \\
&= g(Y,PZ)X(\lambda) - g(X,PZ)Y(\lambda) + B(X,PZ)\eta(Y)\lambda - B(Y,PZ)\eta(X)\lambda \\
&\quad +C(X,PZ)\eta(Y)\lambda - C(Y,PZ)\eta(X)\lambda + \tau(Y)C(X,PZ) - \tau(X)C(Y,PZ),
\end{aligned}$$

replace X by ξ and using (28), (39) we have

$$\begin{aligned}
&g(Y,PZ)\xi(\lambda) - B(Y,PZ)\lambda - g(Y,PZ)\lambda^2 - \tau(\xi)\lambda g(Y,PZ) \\
&\quad = \bar{c}g(Y,PZ),
\end{aligned}$$

replace Y by PZ, we get

$$g(PZ,PZ)\{\xi(\lambda) - \lambda^2 - \tau(\xi)\lambda - \bar{c}\} = \lambda B(PZ,PZ),$$

then by virtue of non degeneracy of $S(TM)$, the theorem follows

Theorem 4.6. Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}(\bar{c}),\bar{g})$ of constant curvature $\bar{c}$ and screen distribution be totally umbilical. If M is also totally umbilical then the function λ satisfies the following partial differential equation

$$X(\lambda) - \lambda\tau(X) - \lambda\rho\eta(X) - \lambda^2 g(X,\xi) - \bar{c}\eta(X) = 0.$$

Proof. Using (31), (39) in (28) we get

$$\begin{aligned}
&\bar{c}g(Y,PZ)\eta(X) - \bar{c}g(X,PZ)\eta(Y) = X(\lambda)g(Y,PZ) - \lambda g(\nabla_X Y,PZ) \\
&\quad -\lambda g(Y,\nabla^*_X PZ) - Y(\lambda)g(X,PZ) + \lambda g(\nabla_Y X,PZ) + \lambda g(X,\nabla^*_Y PZ) \\
&\quad +\tau(Y)\lambda g(X,PZ) - \tau(X)\lambda g(Y,PZ),
\end{aligned}$$

this gives

$$\begin{aligned}
&g(Y,PZ)\{\bar{c}\eta(X) - X(\lambda) + \tau(X)\lambda\} - g(X,PZ)\{\bar{c}\eta(Y) - Y(\lambda) + \tau(Y)\lambda\} \\
&\quad = \lambda\{g(\nabla_Y X,PZ) - g(\nabla_X Y,PZ) + g(X,\nabla^*_Y PZ) - g(Y,\nabla^*_X PZ)\}. \quad (64)
\end{aligned}$$

Now, using (8) and (15), we have

$$g(\nabla_Y X, PZ) = -g(X, \nabla^*_Y PZ) - C(Y, PZ)g(X, \xi) - B(Y, PZ)\eta(X). \quad (65)$$

Therefore, using (65) in (66), we obtain

$$g(Y, PZ)\{\bar{c}\eta(X) - X(\lambda) + \tau(X)\lambda + \lambda^2 g(X, \xi) + \rho\eta(X)\} = g(X, PZ)\{\bar{c}\eta(Y) - Y(\lambda) + \tau(Y)\lambda + \lambda^2 g(Y, \xi) + \rho\eta(Y)\}.$$

Thus using the method given in Theorem 3.4, we have the proof.
Corollary. Let $(M, g, S(TM))$ be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}(\bar{c}), \bar{g})$. If $S(TM)$ is totally geodesic then $\bar{c} = 0$ that is, $\bar{M}$ is a semi-Euclidean space.

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