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Totally Umbilical Submanifolds of Trans-sasakian Manifold

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Abstract
In this article we classify the totally umbilical submanifolds when the ambient manifold is trans-Sasakian.

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1. Introduction

Totally umbilical submanifolds of a Sasakian manifold have been completely classified by Deshmukh and Al-Gwaiz [3].They proved the following

Theorem 1 Let M^n be a complete, simply connected, totally umbilical submanifold of a Sasakian manifold $\bar{M}^{2n+1}$ with mean curvature vector parallel in the normal bundle. Then M is one of the following:

(i) M is totally geodesic,

(ii) M is isometric to a sphere,

(iii) M is homothetic to a Sasakian manifold.

The object of the present paper is to obtain a classification of totally umbilical submanifolds of a trans-Sasakian manifold with parallel mean curvature vector in the normal bundle. In the proof of the main theorem we use the following famous results of [2],[1] and [4]:

Theorem (I) [2] A totally umbilical submanifold M in a real space form $R^{2n+1}(k)$ is either totally geodesic in $R^{2n+1}(k)$ or contained in a hypersphere of an $(n+1)$-dimensional totally geodesic submanifold of $R^{2n+1}(k)$.

Theorem (II) [1] If a Riemannian manifold admits a unit Killing vector field ξ such that $R(X,Y)\xi = \{g(Y,\xi)X - g(X,\xi)Y\}$, then it is a Sasakian manifold.

Axiom (III) [4] A Riemannian manifold M of dimension $m \geq 3$ satisfies the axiom of r-spheres if for each point p in M and for every r-dimensional linear subspace T of T_pM, there exist an r-dimensional totally umbilical submanifold N in M with parallel mean curvature vector field containing p such that $T_pM = T$.

Theorem (IV) [4] A Riemannian manifold of dimension $m \geq 3$ is a real space form if and only if it satisfies the axiom of r-spheres for some $r, 2 \leq r \leq m$.

In this article we prove the folowing theorem of classification of totally umbilical submanifolds of a trans-Sasakian manifold.

Theorem 2 Let M^r be a complete simply connected totally umbilical submanifold of a trans-Sasakian manifold $\bar{M}^{2n+1}$ for all r, $2 \leq r \leq 2n+1$, with mean curvature vector field parallel in the normal bundle ν. Then M^r is one of the following-

(i) M^r is totally geodesic,

(ii) M^r is isometric to a sphere,

(iii) M^r is homothetic to a Sasakian manifold for odd r provided that either X or Y is along ξ_1, $\forall\, X, Y \in T_pM$ and ξ_1 is given by the decomposition $\xi = \xi_1 + \xi_2$, $\xi_1 =$ tangential component of ξ and $\xi_2 = \{e\}^{\perp}-$component of ξ where $\nu = \{e\} \oplus \{e\}^{\perp}$ is the normal bundle of M^r,

(iv) M^r is a complex manifold for even r.

2. Preliminaries

Let $\bar{M}^{2n+1}$ be a $(2n+1)$-dimensional differentiable manifold endowed with the almost contact metric structure (ϕ, ξ, η, g) where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g a Riemannian metric on $\bar{M}$, all these tensor fields satisfying the conditions given below

$$\phi^2 = -I + \eta \otimes \xi \ , \ \phi\xi = 0 \ , \ \eta \circ \phi = 0, \ \eta(\xi) = 1$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \ \eta(X) = g(X, \xi)$$

for any vector fields $X, Y \in T_p\bar{M}$.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a trans-Sasakian structure of type (α, β) if it satisfies

$$(\bar{\nabla}_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}$$

for certain functions α and β on $\bar{M}$, where $\bar{\nabla}$ means the covariant differentiation with respect to g.It is clear that if $\beta = 0$, then the manifold becomes α-Sasakian and if $\alpha = 0$, then the manifold becomes β-Kenmotsu.

We know that a trans-Sasakian manifold is normal and it satisfies

$$\bar{\nabla}_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \quad \forall\, X, \in \chi(\bar{M}). \tag{2.1}$$

Now we know that for totally umbilical submanifolds we have

$$h(X, Y) = g(X, Y)H$$

where H is the mean curvature vector.

Let M^r be a simply connected totally umbilical submanifold of a trans-Sasakian manifold $\bar{M}^{n+1}$ with mean curvature vector field parallel in the normal bundle ν.

3. Proof of the main theorem

Now we know that

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)\ \forall\, X, Y \in \chi(\bar{M})$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \quad \forall\, X \in \chi(\bar{M}), \forall\, N \in \nu$$

or,

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

$$\bar{\nabla}_X N = -g(H,N)X + \nabla^{\perp}_X N.$$

The mean curvature is said to be parallel in the normal bundle ν if $\nabla^{\perp}_X H = 0$. M is said to be totally geodesic if $h(X,Y) = 0\ \forall\ X, Y \in \chi(M)$.
If we set $\|H\|^2 = \mu$, then for the totally umbilical submanifold with mean curvature vector parallel in the normal bundle ν we have $X \cdot \mu = 0$ as $X \cdot \mu = X \cdot \|H\|^2 = X \cdot g(H,H) = 2g(\nabla^{\perp}_X H, H)$ which should be zero for H to be parallel in the normal bundle ν i.e. μ is a constant.
If $\mu \neq 0$, then define a unit vector $e \in \nu$ by setting $H = \sqrt{\mu} e = \|H\|\, e$. The normal bundle ν can then be split into the direct sum $\nu = \{e\} \oplus \{e\}^{\perp}$ where $\{e\}^{\perp}$ is the orthogonal complement of the line sub-bundle $\{e\}$ spanned by e. For each $X \in \chi(M)$, we set

$$\phi X = \psi(X) + A(X)e + P(X)$$

where $\psi(X)$ is the tangential component of ϕX, and $A(X)$ and $P(X)$ are the $\{e\}-$ and $\{e\}^{\perp}-$components of ϕX respectively.

With the help of parallelism of H, we have concluded that μ is constant. Now there arises two cases: either $\mu = 0$ or $\mu \neq 0$.

If $\mu = 0$, then it implies that $H = 0$ which gives $h = 0 \implies M$ is totally umbilical which is the case (i) of the theorem.

Therefore we assume now that $\mu \neq 0$.
By our supposition, we have $\bar{M}^{2n+1}$ as a trans-Sasakian manifold and M^r , a simply connected, complete totally umbilical submanifold.

We have already established the existence of totally umbilical submanifold M of $\bar{M}\ \ \forall p \in M$ and $\forall r$ such that there exists an r-dimensional linear subspace T of $T_p\bar{M}$ which satisfies $T = T_pM$ and $2 \leq r \leq 2n+1$. This shows that $\bar{M}$ satisfies the axiom of r-spheres [4]. But a Riemannian manifold of dimension $2n+1 \geq 3$ satisfies the axiom of r-spheres for some r, $2 \leq r \leq 2n+1$ iff it is a real space form [4].
Also Theorem (I) [4] gives that corresponding to the case $\mu \neq 0$, M may be isometric to a sphere which is the second possibility of the theorem.

Now from equation (2.1) we have

$$\bar{\nabla}_X\xi_1 + \bar{\nabla}_X\xi_2 = -\alpha\psi(X) - \alpha A(X)e - \alpha P(X) + \beta X - \beta g(X,\xi_1)\xi_1 \\ - \beta g(X,\xi_2)\xi_1 - \beta g(X,\xi_1)\xi_2 - \beta g(X,\xi_2)\xi_2$$

or,

$$\nabla_X\xi_1 + g(X,\xi_1)\sqrt{\mu}e + \nabla^{\perp}_X\xi_2 - \sqrt{\mu}g(e,\xi_2)X = -\alpha\psi(X) - \alpha A(X)e - \alpha P(X) + \beta X \\ - \beta g(X,\xi_1)\xi_1 - \beta g(X,\xi_2)\xi_1 - \beta g(X,\xi_1)\xi_2 - \beta g(X,\xi_2)\xi_2.$$

Comparing tangential parts on both sides we have

$$\nabla_X\xi_1 = -\alpha\psi(X) + \beta X - \beta g(X,\xi_1)\xi_1.$$

Therefore

$$g(\nabla_X\xi_1, Y) + g(\nabla_Y\xi_1, X) = g(-\alpha\psi(X) + \beta X - \beta g(X,\xi_1)\xi_1, Y) \\ + g(-\alpha\psi(Y) + \beta Y - \beta g(Y,\xi_1)\xi_1, X).$$

The above equation gives

$$g(\nabla_X\xi_1, Y) + g(\nabla_Y\xi_1, X) = 2\beta g(X,Y) - 2\beta g(Y,\xi_1)g(X,\xi_1).$$

Now the vector field ξ_1 is Killing iff either X or Y is along ξ_1. Let us take $\bar{\xi}_1 = \frac{\xi_1}{\|\xi_1\|}$. This shows that M admits a unit Killing vector field iff either X or Y is along $\bar{\xi}_1$. Now if the ambient space $\bar{M}$ is a space of constant curvature k, then we have

$$\bar{R}(X,Y)Z = k\{g(Z,Y)X - g(Z,X)Y\}$$

for any vector fields X, Y, Z in $\bar{M}$. Hence for any vector fields X, Y, Z in M, $\bar{R}(X,Y)Z$ is tangent to M. Then the equation of Gauss reduces to

$$R(X,Y,Z,W) = g(R(X,Y)Z,W) = k\{g(X,W)g(Y.Z) - g(X,Z)g(Y,W)\} \\ + \bar{g}(h(X,W),h(Y,Z)) - \bar{g}(h(X,Z),h(Y,W))$$

where $\nu = \{e\} \oplus \{e\}^{\perp}$.

Using the totally umbilicity of the submanifold of M^r we conclude that

$$R(X,Y)\xi_1 = k\{g(Y,\xi_1)X - g(X,\xi_1)Y\} + \mu\{g(Y,\xi_1)X - g(X,\xi_1)Y\}$$

or,

$$R(X,Y)\xi_1 = (k+\mu)\{g(Y,\xi_1)X - g(X,\xi_1)Y\}.$$

Now from Theorem (II) [1] M is homothetic to a Sasakian manifold which is the third possibility of the theorem.

If the dimension r of M is even, then naturally it will be a complex manifold [Page-369; 5]. Hence the theorem is proved.

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