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Semi-Slant Submanifolds of a Nearly Kaehler Manifold

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Abstract
Warped product manifolds provide a natural frame work for time dependent mechanical system and have applications in Physics (cf.[2],[11],[13]). The studies on warped product manifolds with extrinsic geometric point of view intensified after B.Y.Chen's work on CR-warped product submanifolds of Kaehler manifolds (cf.[8],[9]). He investigated the existence of CR-submanifolds of a Kaehler manifold which are warped product manifolds and established a characterization under which a CR-submanifold reduces to a CR-warped product. Subsequently, similar studies are done in nearly Kaehler manifolds as well (cf.[14],[15],[18]). The present note is devoted to study semi-slant submanifolds of a nearly Kaehler manifold. In particular, a characterization is worked out under which a semi-slant submanifold of a nearly Kaehler manifold reduces to a semi-slant warped product submanifold. The results presented improve and extend the corresponding results of Kaehlerian settings.

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1. Introduction

R.L.Bishop and B.O'Neill [4], while investigating manifolds of negative sectional curvatures generalized the notion of (Riemannian) product metric by homothetically warping the product metric on to the fibers. To be more precise, if (M_1, g_1) and (M_2, g_2) are Riemannian manifolds and f is a smooth function on M_1, then the warped metric g on $M_1 \times M_2$ is defined as $g = g_1 + f^2 g_2$. The manifold $(M_1 \times M_2, g)$, denoted by $M_1 \times_f M_2$ is known as warped product of M_1 and M_2. If the warping function f is just a constant, the warped product metric reduces to a Riemannian product. Bishop and O'Neill obtained some important properties of these manifolds with intrinsic geometric point of view. Warped product manifolds provide an excellent setting to model space-time near black holes or bodies with high gravitational fields (cf.[13]). The realization of these applications paved way for the study of warped product manifolds with extrinsic geometric point of view.

It is known that a submanifold of an almost Hermitian manifold $\bar{M}$ is a CR-product if it is locally a Riemannian product of a holomorphic submanifold M_T and a totally real submanifold $M_\perp$ of $\bar{M}$. These submanifolds are CR-submanifolds in the sense of A.Bejancu [3]. However, a CR-submanifold is not a CR-product unless the two canonical distributions D and $D^\perp$ on M are parallel. As a first step in the study of warped product manifolds with extrinsic geometric point of view, B.Y.Chen [8] considered CR-submanifolds of a Kaehler manifold as warped product manifolds i.e., submanifolds of the types $M_T \times_f M_\perp$ and $M_\perp \times_f M_T$, termed as CR-warped product and warped product CR-submanifolds respectively. He found that warped product CR-submanifolds in a Kaehler manifold are trivial i.e., these submanifolds are simply CR-products. However, non-trivial CR-warped products do exist in a Kaehler manifold. The non-existence of a larger class of warped product submanifolds namely the submanifolds of the type $M_0 \times_f M_T$, in a Kaehler manifold (as well as in a nearly Kaehler manifold) $\bar{M}$, where M_0 is an arbitrary submanifold of $\bar{M}$, is established by V.A.Khan et.al. (cf. [1],[14]). A Kaehler manifold does not admit even warped product submanifold of the type $M_T \times_f M_0$ (cf. [1],[17]). That means, the only non-trivial warped product submanifolds in a Kaehler manifold with one of the factors a holomorphic submanifold is a CR-warped product submanifold. On the other hand, a nearly Kaehler manifold may admit such warped products (see Example 4.1).

B.Y.Chen [5] proved that a CR-submanifold of a Kaehler manifold is a CR-product if and only if $A_{JD^\perp}D = 0$. Since a CR-product is a special case of CR-warped product, it is natural to seek conditions under which a CR-submanifold reduces to a CR-warped product. To this end, generalizing his earlier result, Chen [8] showed that a CR-submanifold of a Kaehler manifold is a CR-warped product if and only if $A_{JZ}X = -(JX\alpha)Z$, $X \in D$ and $Z \in D^\perp$ where α is a smooth function on M such that $W\alpha = 0$ for each $W \in D^\perp$. Similar characterizations were obtained for CR-warped products of nearly Kaehler manifolds by V.A.Khan et. al [15] and B. Sahin [18]. Since CR-products in S^6 are non-existent, K.Sekigawa [19] explored CR-warped products in S^6 and obtained an example of the same. One of the next step forward is to look for semi-slant warped products (a more general class of warped products than the class of CR-warped products) in nearly Kaehler manifolds. Our study of these submanifolds has led us to obtain a characterization under which a semi-slant submanifold of a nearly Kaehler manifold reduces to semi-slant warped product, thereby generalize the result of Chen [5],[8], V.A.Khan et. al [15] and Sahin [18].

2. Preliminaries

Let $(\bar{M}, J, g)$ be an almost Hermitian manifold with an almost complex structure J

and Hermitian metric g. The Nijenhuis tensor S of J is defined as :

$$S(U,V) = [JU,JV] - [U,V] - J[JU,V] - J[U,JV], \quad U,V \in T\bar{M}. \tag{2.1}$$

Let $\bar{\nabla}$ be Levi-Civita connection on $\bar{M}$. If J is parallel with respect to $\bar{\nabla}$, i.e. $\bar{\nabla}J = 0$, then $\bar{M}$ is a Kaehler manifold. A more general structure on $\bar{M}$, known as nearly Kaehler structure is defined by a weaker condition namely

$$(\bar{\nabla}_U J)U = 0 \tag{2.2}$$

or equivalently

$$(\bar{\nabla}_U J)V + (\bar{\nabla}_V J)U = 0.$$

A necessary and sufficient condition for a nearly Kaehler manifold to be a Kaehler manifold is the vanishing of the Nijenhuis tensor of J. Any four dimensional nearly Kaehler manifold is a Kaehler manifold. A typical example of a nearly Kaehler, non-Kaehler manifold is the six dimensional sphere S^6. It has an almost complex structure J defined by the vector cross product in the space of purely imaginary Cayley numbers (cf.[10]). This almost complex structure is not integrable and satisfies (2.2). The Nijenhuis tensor S of J on $\bar{M}$ satisfies:

$$S(U,V) = -4J(\bar{\nabla}_U J)V \tag{2.3}$$

Let ∇ and $\nabla^{\perp}$ be the induced Levi-Civita connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of a submanifold M of $\bar{M}$. Then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U,V) \tag{2.4}$$

and

$$\bar{\nabla}_U \xi = -A_{\xi}U + \nabla^{\perp}_U \xi \tag{2.5}$$

for $U,V \in TM$ and $\xi \in T^{\perp}M$; where h is the second fundamental form and A_{ξ}, the shape operator (corresponding to the normal vector field ξ) of the immersion of M into $\bar{M}$. The two are related by

$$g(A_{\xi}U,V) = g(h(U,V),\xi) \tag{2.6}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the one induced on M.

For any $U \in TM$, we put

$$PU = tan(JU) \ \ and \ \ FU = nor(JU), \tag{2.7}$$

where tan_x and nor_x are the natural projections associated to the direct sum decomposition

$$T_x\bar{M} = T_xM \oplus T^{\perp}_xM, \ \ x \in M$$

Similarly for $\xi \in T^{\perp}M$, we put

$$t\xi = tan(J\xi) \;\; and \;\; f\xi = nor(J\xi) \tag{2.8}$$

That is P (resp. f) is a (1,1) tensor field on TM (resp. $T^{\perp}M$) whereas t (resp. F) is a tangential (resp. normal) valued 1-form on $T^{\perp}M$ (resp. TM).

The covariant differentiations of the tensor fields P, F, t and f are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.9}$$

$$(\bar{\nabla}_U F)V = \nabla_U^{\perp} FV - F\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla_U^{\perp}\xi, \tag{2.11}$$

$$(\bar{\nabla}_U f)\xi = \nabla_U^{\perp} f\xi - f\nabla_U^{\perp}\xi. \tag{2.12}$$

Furthermore, if we denote the tangential and normal parts of $(\bar{\nabla}_U J)V$ by $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ respectively then on making use of (2.4),(2.5) and (2.7)-(2.12), we obtain

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U,V), \tag{2.13}$$

$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + h(U,PV) - fh(U,V). \tag{2.14}$$

A CR-submanifold M of an almost Hermitian manifold $\bar{M}$ is a submanifold endowed with two orthogonal complementary distributions D and $D^{\perp}$ such that D is holomorphic i.e., $JD_x = D_x$ and $D^{\perp}$ is totally real i.e., $JD_x^{\perp} \subseteq T_x^{\perp}M$ for each $x \in M$, and $TM = D \oplus D^{\perp}$. A CR-submanifold is holomorphic (resp. totally real) if $D^{\perp} = 0$ (resp. $D = 0$). A CR-submanifold is proper if both the distributions D and $D^{\perp}$ on M are non-trivial.

It is easy to observe that the angle between JX and PX is zero for each vector field $X \in D$ whereas JZ, for each $Z \in D^{\perp}$, makes an angle $\frac{\pi}{2}$ from $D^{\perp}$ (threfore, orthogonal to the tangent bundle TM of the submanifold M). This stand point gives rise to a more general distribution, namely slant distribution on a submanifold of an almost Hermition manifold.

A distribution D^{θ} on a submanifold M of an almost Hermitian manifold is called a slant distribution if the wirtinger angle $\theta(X) \in [0, \pi/2]$ between JX and D_x^{θ} has the same value θ for each $x \in M$ and $X \in T_x(M)$, $X \neq 0$. A submanifold M is called a slant submanifold if the tangent bundle TM is slant. Holomorphic and totally real submanifolds are special cases of slant submanifolds with wirtinger angle 0 and $\pi/2$ respectively. A slant submanifold is called proper slant if it is neither holomorphic nor totally real (cf.[7]).

If M is a slant submanifold of an almost Hermitian manifold $\bar{M}$, then we have

$$P^2 = -cos^2\theta I, \tag{2.15}$$

where θ is the wirtinger angle of M in $\bar{M}$. This gives

$$g(PU, PV) = cos^2\theta g(U, V), \tag{2.16}$$

$$g(FU, FV) = sin^2\theta g(U, V), \tag{2.17}$$

for U, V tangent to M [7].

3. Semi-slant submanifolds

As a natural generalization of CR-submanifolds, N. Papaghiuc [16] introduced the notion of semi-slant submanifolds as:

A submanifold M of an almost Hermitian manifold is called a semi-slant submanifold if it is endowed with two orthogonal complementary distributions D and D^θ such that D is holomorphic and D^θ is slant. A semi-slant submanifold M is a CR-submanifold if the slant distribution D^θ on M is totally real, i.e. $\theta = \pi/2$ whereas a semi-slant submanifold reduces to a slant submanifold if $D = \{0\}$. A semi-slant submanifold is proper if $\theta \neq \pi/2$.

It follows straight away from the definition that for a semi-slant submanifold M of an almost Hermitian manifold $\bar{M}$, the tangent bundle TM and the normal bundle $T^\perp M$ are decomposed as

$$TM = D \oplus D^\theta \tag{3.1}$$

and

$$T^\perp M = FD^\theta \oplus \mu \tag{3.2}$$

where μ is the orthogonal complementary distribution to FD^θ in $T^\perp M$ and is invariant under J. This means $J\xi = f\xi$ for each $\xi \in \mu$ whereas $f\xi \in FD^\theta$ for each $\xi \in FD^\theta$. Moreover, following are some other easy observations

$$\left.\begin{array}{ll} (a)\ FD = \{0\}, & (b)\ PD = D, \\ (c)\ PD^\theta \subseteq D^\theta, & (d)\ t(T^\perp M) = D^\theta. \end{array}\right\} \tag{3.3}$$

In terms of P, F, t and f, we have

$$\left.\begin{array}{ll} (e)\ P^2 + tF = -I, & (f)\ f^2 + Ft = -I, \\ (g)\ FP + fF = 0, & (h)\ tf + Pt = 0. \end{array}\right\} \tag{3.4}$$

Throughout this section, M denotes a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$. With regard to the integrability conditions of the distributions in this setting, we have:

Theorem 3.1. Let M be a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then the following are equivalent

(a) The holomorphic distribution D is involutive.

(b) $h(X, JY) = h(JX, Y)$, and $Q_X Y = 0$.

(c) $2g((\bar{\nabla}_X J)Y, JZ) = g(\nabla_X PY - \nabla_Y PX, PZ) + g(h(X, PY) - h(Y, PX), FZ)$,

for each $X, Y \in D$ and $Z \in D^\theta$

Proof. By formula (2.1),

$$(S(X,Y))^\perp = -F([JX,Y] + [X, JY]),$$

where $\perp$ stands for the normal part. On the other hand, by formula (2.3)

$$(S(X,Y))^\perp = 4Q_X JY.$$

From the last two equations,

$$4Q_X JY = -F([JX,Y] + [X,JY]). \tag{3.5}$$

Now, by virtue of nearly Kaehler condition, $Q_X Y - Q_Y X = 2Q_X Y$. Using this fact in formulae (2.14) and (2.10), we get

$$2Q_X Y = F[Y,X] + h(X,JY) - h(Y.JX). \tag{3.6}$$

If D is involutive, by (3.5), $Q_X Y = 0$. Thus assuming D involutine on M, we obtain from (3.6) that $h(X, JY) = h(Y, JX)$. Hence (a) implies (b). Conversely, if $h(X, JY) = h(Y, JX)$ and $Q_X Y = 0$, then by (3.6), $[X, Y] \in D$. That is, D is involutive proving the equivalence of (a) and (b).

For the eqivalence of (a) and (c), consider $g([X,Y], Z)$ for $X, Y \in D$ and $Z \in D^\theta$,

$$\begin{aligned} g([X,Y],Z) &= g(\bar{\nabla}_X Y - \bar{\nabla}_Y X, Z) \\ &= g(J\bar{\nabla}_X Y, JZ) - g(J\bar{\nabla}_Y X, JZ) \\ &= g(\bar{\nabla}_X JY - (\bar{\nabla}_X J)Y, JZ) - g(\bar{\nabla}_Y JX - (\bar{\nabla}_Y J)X, JZ). \end{aligned}$$

Making use of Gauss formula and nearly Kaehler condition, the equation reduces to

$$\begin{aligned} g([X,Y],Z) =& g(\nabla_X JY - \nabla_Y JX, PZ) + g(h(X,JY) - h(Y,JX), FZ) \\ & - 2g((\bar{\nabla}_X J)Y, JZ) \end{aligned}$$

Thus D is involutive if and only if

$$\begin{aligned} 2g((\bar{\nabla}_X J)Y, JZ) &= g(\nabla_X PY - \nabla_Y PX, PZ) \\ &\quad + g(h(X,PY) - h(Y,PX), FZ). \end{aligned} \tag{3.7}$$

Hence (a),(b) and (c) are equivalent.

In particular, if M is a CR-submanifold, the integrability condition (3.7) reduces to:

$$2g((\bar{\nabla}_X J)Y, JZ) = g(h(X, PY) - h(Y, PX). FZ).$$

It can be deduced from the above equation that the holomorphic distribution on a CR-submanifold of a Kaehler manifold is involutive if and only if

$$g(h(X, JY) - h(JX, Y), FZ) = 0.$$

This condition is worked out by B.Y.Chen in [5].

Theorem 3.2. The slant distribution D^θ on a semi-slant submanifold M of a nearly Kaehler manifold is involutive if and only if

$$2g(\nabla_W Z. X) = g(\nabla_Z PW + \nabla_W PZ - A_{FZ}W - A_{FW}Z, PX), \qquad (3.8)$$

for each $X \in D$ and $Z, W \in D^\theta$

Proof. Consider, $g([Z, W], X)$. We can write

$$\begin{aligned} g([Z, W]. X) &= g(\bar{\nabla}_Z W - \bar{\nabla}_W Z, X) = g(J\bar{\nabla}_Z W - J\bar{\nabla}_W Z, JX) \\ &= g(\bar{\nabla}_Z JW - \bar{\nabla}_W JZ - (\bar{\nabla}_Z J)W + g(\bar{\nabla}_W J)Z, JX) \end{aligned}$$

On making use of the formulae (2.4),(2.5), (2.7) and the nearly Kaehler condition, it can be deduced from the above equation that

$$\begin{aligned} g([Z. W], X) =& g(\nabla_Z PW - \nabla_W PZ + A_{FZ}W - A_{FW}Z, JX) \\ &+ 2g((\bar{\nabla}_W J)Z, JX). \end{aligned}$$

Further as

$$g((\bar{\nabla}_W J)Z, JX) = g(\nabla_W PZ - A_{FZ}W, JX) - g(\nabla_W Z, X),$$

the above equation takes the form

$$\begin{aligned} g([Z, W], X) = & g(\nabla_Z PW + \nabla_W PZ - A_{FZ}W - A_{FW}Z, JX) \\ & - 2g(\nabla_W Z, X) \end{aligned}$$

The assertion follows from the above formula.

In particular, if M is a CR-submanifold, the integrability condition (3.8) takes the following form. The totally real distribution on a CR-submanifold of a nearly Kaehler manifold is involutive if and only if

$$2g(\nabla_W Z, X) = -g(A_{FZ}W + A_{FW}Z, JX).$$

It is known that the totally real distribution on a CR-submanifold of a Kaehler manifold is involutive (cf. [5]). With regard to the geometry of its leaves, we have

Corollary 3.1. The leaves of totally real distribution $D^{\perp}$ on a CR-submanifold M of a Kaehler manifold are totally geodesic in M if and only if

$$g(h(D, D^{\perp}), JD^{\perp}) = 0$$

Proof. In the given setting, $A_{JZ}W = A_{JW}Z$ (cf. [5]). Using this fact together with formula (2.6) in the last equation, we get

$$g(\nabla_W Z, X) = -g(h(JX, Z), JW)$$

for each $X \in D$ and $Z, W \in D^{\perp}$. The assertion follows from the above equation. This fact is proved by Chen in [5].

With regard to the parallelism of D, we have

Theorem 3.3. Let M be a proper semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then the holomorphic distribution D is parallel if and only if

$$(\bar{\nabla}_X P)Y \in D$$

for each $X, Y \in D$.

Proof. If D is parallel, then by formula (2.9), $(\bar{\nabla}_X P)Y \in D$. Conversely, suppose that $(\bar{\nabla}_X P)Y \in D$ for all $X, Y \in D$, then by virtue of (2.9),

$$g(\nabla_X PY - P\nabla_X Y, Z) = 0 \quad \text{for all} \quad Z \in D^{\theta}$$

or

$$g(\nabla_X PY, Z) + g(\nabla_X Y, PZ) = 0 \tag{3.9}$$

Replacing Y by PY and Z by PZ in the above equation, yields

$$g(\nabla_X Y, PZ) + cos^2\theta g(\nabla_X PY, Z) = 0 \tag{3.10}$$

Subtracting (3.10) from (3.9), we obtain

$$Sin^2\theta \; g(\nabla_X PY, Z) = 0$$

As M is a proper semi-slant submanifold, we have

$$g(\nabla_X PY, Z) = 0,$$

for each $X, Y \in D$ and $Z \in D^{\theta}$. That is, D is parallel.

4. Semi-slant submanifolds as warped product submanifolds

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively. Then the product manifold $M = M_1 \times M_2$ is a Riemannian manifold endowed with the Riemannian metric g defined as

$$g(U, V) = g_1(d\pi_1 U, d\pi_1 V) + g_2(d\pi_2 U, d\pi_2 V) \tag{4.1}$$

where $\pi_i (i = 1, 2)$ are the projection maps of M onto M_1 and M_2 respectively and $d\pi_i (i = 1, 2)$ are their differentials. Generalizing the product metric (4.1) warped product of M_1 and M_2 is defined as:

Let f be a positive differentiable function on M_1. Then the warped product $M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ endowed with Riemannian metric g given by

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2) \tag{4.2}$$

More explicitly, if U is tangent to $M = M_1 \times_f M_2$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2.$$

The function f, in this case is known as the warping function (cf.[4]). If the warping function f is just a constant, the warped product is simply a Riemannian product, known as a trivial warped product.

Few important observations and formulae revealing some geometric aspects of a warped product manifold are obtained by Bishop and O'Neill and are stated as under:

Proposition 4.1. [4] Let $M = M_1 \times_f M_2$ be a warped product manifold. If $X, Y \in TM_1$ and $Z, W \in TM_2$, then

(i) $\nabla_X Y \in TM_1$,

(ii) $\nabla_X Z = \nabla_Z X = (Xlnf)Z$,

(iii) $nor(\nabla_Z W) = -g(Z, W)\nabla lnf$

where $nor(\nabla_Z W)$ denotes the component of $\nabla_Z W$ in TM_1 and ∇f is the gradient of f defined as

$$g(\nabla f, U) = Uf \tag{4.3}$$

for any $U \in TM$.

Corollary 4.1. On a warped product manifold $M = M_1 \times_f M_2$,

(i) M_1 is totally geodesic in M

(ii) M_2 is totally umbilical in M.

Throughout, we denote by M_T, a holomorphic and M_θ, a slant submanifold (with wirtinget angle θ) of an almost Hermitian manifold $\bar{M}$.

A semi-slant submanifold of $\bar{M}$ which is a warped product manifold has the form $M_T \times_f M_\theta$ or $M_\theta \times_f M_T$. If $\theta = \frac{\pi}{2}$, these warped product submanifolds reduce to CR-warped product and warped product CR-submanifolds respectively. The existence of these submanifolds in Kaehler manifold is explored by B.Y.Chen

[8] and in nearly Kaehler manifolds by V.A.Khan et. al [15]. For proper semi-slant warped product submanifolds of Kaehler manifolds, B.Sahin established the following:

Theorem 4.1. **[17]** Let $\bar{M}$ be a Kaehler manifold. Then there do not exist warped product submanifolds $M_\theta \times_f M_T$ and $M_T \times_f M_\theta$ in $\bar{M}$ such that M_θ is a proper slant submanifold and M_T is a holomorphic submanifold.

One of the next steps of extending the study is to consider warped product submanifolds with one of the factors a holomorphic submanifold and the other not necessarily slant. These submanifolds are generic in the sense of B.Y.Chen [6]. Thus if M_0 is a submanifold of $\bar{M}$, then $M_0 \times_f M_T$ and $M_T \times_f M_0$ are generic warped product submanifolds of $\bar{M}$. Generic warped product submanifolds of a Kaehler manifold are trivial (cf. [1]). So far as generic warped products of nearly Kaehler manifolds are concerned, V.A.Khan et. al [14] obtained.

Theorem 4.2.[14] Let $\bar{M}$ be a nearly Kaehler manifold and $M = M_0 \times_f M_T$ a warped product submanifold of $\bar{M}$ with M_0 and M_T a Riemannian and a holomorphic submanifold of $\bar{M}$. Then M is trivial i.e., M is locally a Riemannian product of M_0 and M_T.

As an immediate consequence of the above theorem non-trivial semi-slant warped product of the type $M_\theta \times_f M_T$ are non-existent in a nearly Kaehler manifold. However, warped products of the form $M_T \times_f M_\theta$ do exist in nearly Kaehler manifolds e.g.,

Example 4.1.[19] Consider CR-submanifolds in S^6, which is the image of $S^2 \times S^1$ into S^6 under the immersion ψ defined as:

$$\begin{aligned}\psi(y,z) &= \psi((y_2,y_4,y_6),e^{it})\\ &= (y_2\cos t)e_2 - (y_2\sin t)e_3 + (y_4\cos 2t)e_4 + (y_4\sin 2t)e_5\\ &\quad + (y_6\cos t)e_6 - (y_6\sin t)e_7\end{aligned}$$

for $y = (y_2,y_4,y_6) \in S^2$ and $z = e^{it} \in S^1$, $t \in \mathbb{R}$. Then consider the tangent vectors

$$\begin{aligned}Z_1 &= -(y_2\sin t)e_2 - (y_2\cos t)e_3 - (2y_4\sin 2t)e_4 + (2y_4\cos 2t)e_5\\ &\quad - (y_6\sin t)e_6 - (y_6\cos t)e_7\\ Z_2 &= (y_6\cos t)e_2 - (y_6\sin t)e_3 - (y_2\cos t)e_6 - (y_2\sin t)e_7\\ Z_3 &= (y_6\cos 2t)e_4 + (y_6\sin 2t)e_5 - (y_4\cos t)e_6 - (y_4\sin t)e_7\end{aligned}$$

Let us denote $span\{Z_2,Z_3\}$ by D and $D^\perp = span\{Z_1\}$. Then we can derive that D is integrable. Denoting the integral manifolds of D and $D^\perp$ by M_T and $M_\perp$ respectively, the induced metric tensor is given by:

$$\begin{aligned}ds^2 =&(y_6^2+y_2^2)dy_2^2 + y_2y_4dy_2dy_4 + (y_6^2+y_4^2)dy_4^2 + (1+3y_4^2)dt^2\\ =&g_{M_T} + (1+3y_4^2)g_{M_\perp}\end{aligned}$$

Thus it follows that $M = M_T \times_f M_\theta$ is a CR-warped product submanifold of S^6 with warping function $f = \sqrt{(1+3y_4^2)}$.

Our aim in this section is to prove the following

Theorem 4.3. A proper semi-slant submanifold M of a nearly Kaehler manifold $\bar{M}$ is locally a semi-slant warped product if and only if $(\bar{\nabla}_X P)Y \in D$ for each $X, Y \in D$ such that

$$A_{FZ}X = -[(PX\alpha)Z + \frac{(X\alpha)}{3}PZ], \tag{4.4}$$

and

$$2g(\nabla_W Z, X) = g(\nabla_Z PW + \nabla_W PZ, X) - 2(X\alpha)g(Z, W) \tag{4.5}$$

for each $X \in D$ and $Z, W \in D^\theta$, where α is a C^∞-function on M with $W\alpha = 0$, for each $W \in D^\theta$.

Let $M = M_T \times_f M_\theta$ be a semi-slant warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then, as M_T is totally geodesic in M, $(\bar{\nabla}_X P)Y \in D$ for each $X, Y \in D$ and therefore by formula (2.13),

$$\begin{aligned} g(\mathcal{P}_X Y, Z) &= -g(th(X,Y), Z) \\ &= g(h(X,Y), FZ) \end{aligned}$$

The left hand side is skew symmetric whereas the right hand side is symmetric in X and Y in the above equation. That means

$$g(h(X,Y), FZ) = g(\mathcal{P}_X Y, Z) = 0, \tag{4.6}$$

for each $X, Y \in D$ and $Z \in D^\theta$.

Proof of Theorem 4.3. If $M = M_T \times_f M_\theta$ is a semi-slant warped product submanifold of $\bar{M}$, then by Corollary 4.1, M_T is totally geodesic in M and therefore by (2.9) $(\bar{\nabla}_X P)Y \in D$.

Further, by formula (ii) in Proposition 4.1

$$g(\nabla_W Z, X) = -g(\nabla_W X, Z) = -X ln f g(Z, W).$$

On the other hand using the same formula, it follows that $g(\nabla_Z PW + \nabla_W PZ, X) = 0$. This proves that for semi-slant warped product submanifolds of a nearly Kaehler manifold, formula (4.5) holds.

Further, by (4.6), $g(A_{FZ}X, Y) = 0$. That means

$$A_{FZ}X \in D^\theta, \quad X, Y \in D, \; Z \in D^\theta \tag{4.7}$$

Now, making use of nearly Kaehler condition and formula (2.13), we obtain that

$$0 = (\bar{\nabla}_X P)Z + (\bar{\nabla}_Z P)X - 2th(X,Z) - A_{FZ}X.$$

Using (2.9) and the formula (ii) in Proposition 4.1, we deduce that

$$(\bar{\nabla}_X P)Z = 0 \text{ and } (\bar{\nabla}_Z P)X = (PXlnf)Z - (Xlnf)PZ.$$

On substituting the above values, the last equation takes the form

$$(PXlnf)Z - (Xlnf)PZ = 2th(X,Z) + A_{FZ}X.$$

Taking product with $W \in D^\theta$ in the above equation, gives

$$(PXlnf)g(Z,W) - (Xlnf)g(PZ,W) = -2g(h(X,Z),FW) + g(h(X,W),FZ),$$

which on simplifying yields

$$3g(h(X,Z),FW) = -3(PXlnf)g(Z,W) + (Xlnf)g(PZ,W).$$

Interchanging Z and W, we get

$$g(h(X,W),FZ) = -\left[(\frac{Xlnf}{3})g(PZ,W) + (PXlnf)g(Z,W)\right]$$

or,

$$g(A_{FZ}X,W) = -\left[(\frac{Xlnf}{3})g(PZ,W) + (PXlnf)g(Z,W)\right].$$

As it is observed in (4.7) that $A_{FZ}X \in D^\theta$, we deduce from the above equation that

$$A_{FZ}X = -\left[(\frac{Xlnf}{3})PZ + (PXlnf)Z\right].$$

This establishes (4.4).

Conversely, suppose that M is a semi-slant submanifold of a nearly Kaehler manifold $\bar{M}$ with $(\bar{\nabla}_X P)Y \in D$, for each $X, Y \in D$, and there exist a C^∞- function α on M such that $W\alpha = 0$ for each $W \in D^\theta$ such that (4.4) and (4.5) hold.

The condition $(\bar{\nabla}_X P)Y \in D$, in view of the Theorem 3.3 implies that D is parallel. That is, D is involutive and its leaves are totally geodesic in M.

On the other hand, the orthogonal complementary distribution D^θ of D on M is involutive by virtue of equation (4.5).

Let M_θ be a leaf of D^θ and h^0 be the second fundamental form of M_θ into M. Then for any $X \in D$ and $Z, W \in D^\theta$, we have

$$\begin{aligned} g(h^0(Z,W),JX) &= g(\nabla_Z W, JX) \\ &= g(\bar{\nabla}_Z W, JX) \\ &= -g(J\bar{\nabla}_Z W, X) \\ &= g((\bar{\nabla}_Z J)W, X) - g(\bar{\nabla}_Z JW, X) \qquad (4.8) \end{aligned}$$

Thus, we have,

$$g(h^0(Z,W),JX) = g((\bar{\nabla}_Z J)W,X) - g(\nabla_Z PW,X) + g(A_{FW}Z,X). \quad (4.9)$$

On interchanging Z and W and adding the obtained equation in (4.9) while using nearly Kaehler condition, we get

$$\begin{aligned} 2g(h^0(Z,W),JX) =& g(A_{FW}X,Z) + g(A_{FZ}X,W) \\ &- g(\nabla_Z PW + \nabla_W PZ, X). \end{aligned}$$

Making use of (4.4), (4.5) and (4.8) in the above equation, we get

$$g(h^0(Z,W),X+JX) = -[(X\alpha) + (JX\alpha)]g(Z,W)$$

which on applying formula (4.3) takes the form:

$$g(h^0(Z,W),X+JX) = -g(\nabla\alpha, X+JX)g(Z,W).$$

Hence,

$$g(h^0(Z,W),X) = -g(\nabla\alpha,X)g(Z,W)$$

for each $X \in D$ and $Z, W \in D^\theta$. Hence, we have

$$h^0(Z,W) = -g(Z,W)\nabla\alpha.$$

This implies that, the leaves of D^θ are totally umbilical in M. Further, the condition $W\alpha = 0$ implies that the leaves of D^θ are extrinsic spheres in M.

Hence by the result of Hiepko [12] which states that "*If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_1 \oplus E_0$ of non-trivial vector sub-bundles such that E_1 is auto-parallel and its orthogonal complement E_0 is spherical, then the manifold M is locally isometric to the warped product $N_1 \times_f N_0$*", we conclude that M is locally a semi-slant warped product submanifold of $\bar{M}$.

Corollary 4.2. A CR-submanifold of a nearly Kaehler manifold with $(\bar{\nabla}_X P)Y \in D$ $X, Y \in D$, is a CR-warped product submanifold if and only if there exist a smooth function α on M with $W\alpha = 0$ for all $W \in D^\perp$ such that the following conditions are satisfied:

$$A_{JZ}X = -(JX\alpha)Z, \quad (4.10)$$

$$g(\nabla_W Z, X) = -(X\alpha)g(Z,W), \quad (4.11)$$

for each $X \in D$ and $Z, W \in D^\perp$.

Since $PZ = 0$ for each $Z \in D^\perp$, (4.10) and (4.11) follow immediately from (4.4) and (4.5) respectively.

Note. B.Sahin [18] proved a similar characterization under different set of conditions whereas V.A.Khan et. al [15] obtained the characterization with only condition (4.10) while assuming the integrabilities of the two distributions on M.

In the setting of a CR-submanifold M of a Kaehler manifold $\bar{M}$, the conditions (4.4) and (4.5) reduce to (4.10) and (4.11) respectively. Condition (4.10) implies that $h(X,Y) \in \mu$ for each $X, Y \in D$ which means D is parallel i.e., D is integrable and its leaves are totally geodesic in M. In particular $(\bar{\nabla}_X P)Y \in D$. Taking product with $W \in D^\perp$ in (4.10) gives

$$g(A_{JZ}X, W) = -(JX\alpha)g(Z,W). \tag{4.12}$$

Further, as M is assumed to be a CR-submanifold of $\bar{M}$, we have $g(\nabla_U Z, X) = -g(A_{JZ}U, JX)$, for each $U \in TM, X \in D$ and $Z \in D^\perp$ (cf.[5]). In particular, for $U = W \in D^\perp$, the above formula takes the form:

$$g(\nabla_W Z, X) = -g(A_{JZ}W, JX) = -g(A_{JZ}JX, W).$$

The right hand side of the above equation, on using (4.12) reduces to $-(X\alpha)g(Z,W)$. It is known that the totally real distribution $D^\perp$ on a CR-submanifold of a Kaehler manifold is involutive (cf.[5]). Let $M_\perp$ be a leaf of $D^\perp$ and h^0, the second fundamental form of $M_\perp$ in M. Then the last equation may be written as:

$$g(h^0(Z,W), X) = -(X\alpha)g(Z,W).$$

Hence we get

$$h^0(Z,W) = -g(Z,W)\nabla\alpha.$$

Further as $W\alpha = 0$ for each $W \in D^\perp$, we deduce that $M_\perp$ is an extrinsic sphere in M. Hence, by the Theorem of Hiepko, M is locally isometric to a warped product manifold $M_T \times_f M_\perp$. Conversely, if M is a CR-warped product submanifold of a Kaehler manifold $\bar{M}$, then it is straightforward to verify (4.10). Hence we conclude that

Corollary 4.3. Let M be a CR-submanifold of a Kaehler manifold $\bar{M}$. Then M is a CR-warped product submanifold if and only if

$$A_{JZ}X = -(JX\alpha)Z,$$

for each $X \in D, Z \in D^\perp$, where α is a smooth function on M such that $W\alpha = 0$ for each $W \in D^\perp$.

Note. The above corollary was proved by Chen [8] as a characterization for the existence of a CR-warped product submanifold in a Kaehler manifold. Thus, Theorem 4.3 provides a generalization of Chen's Theorem.

References

[1] F.R.Al-Solamy and V.A.Khan, *Non-existence of non-trivial generic warped product in Kaehler manifolds,* Note di Mathematica., 28 (2)(2008), 63-68.

[2] J.K.Beem, P.Ehrlich and T.G.Powell, *warped product manifolds in Relativity in selected studies,* North-Holland, Amsterdam, 1982.

[3] A.Bejancu, *CR-submanifold of a Kaehler manifold I,* Proc. Amer. Math. Soc., 89 (1978), 135-142.

[4] R.L.Bishop and B.O'Neill, *Manifolds of negative curvature,* Trans. Amer. Math. Soc., 145 (1969), 1-49.

[5] B.Y.Chen, *CR-submanifolds of a Kaehler Manifold I,* J. Diff. Geom., 16 (1981), 305-322.

[6] B.Y.Chen, *Differential Geometry of a real submanifold in a Kaehler manifold,* Monatsh. Math., 91 (1981), 257 -275.

[7] B.Y.Chen, *Geometry of slant-submanifolds,* Katholieke Universiteit Leuven, Leuven, (1990).

[8] B.Y.Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds I,* Monatsh. Math., 133 (2001), 177-195.

[9] B.Y.Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II,* Monatsh. Math., 134 (2001), 103-119.

[10] T.Fukami and S. Ishihara, *Almost Hermitian Structure on S^6,* Tohoku Math. J., 7(3) (1955), 151 -156.

[11] S.W.Hawkings and G.F.R. Ellis, *The large scale structure of space-time,* Cambridge Univ. Press, Cambridge, (1973).

[12] S.Hiepko, *Eine Innere Kennzeichung der verzerrten Produkte,* Math. Ann. 241 (1979), 209 -215.

[13] S.T.Hong, *Warped products and black holes,* Nuovo Cim, B120 (2005), 1227 -1234.

[14] V.A.Khan and K.A.Khan, *Generic warped product submanifolds of nearly Kaehler manifolds,* Contributions to Algebra and Geom., 50(2) (2009), 337-352.

[15] V.A.Khan, K.A.Khan and Siraj-Uddin, *Warped product CR-submanifolds in nearly Kaehler manifolds,* SUT Journal of Mathematics, 43 (2) (2007), 201-213.

[16] N.Papaghiuc, *Semi-slant submanifolds of Kaehler manifold,* An. St. Univ. AI. I. Cuza. Iasi, 40 (1994), 55-61.

[17] B.Sahin, *Non-existence of warped product semi-slant submanifolds of Kaehler manifold,* Geom. Dedicata., 1172 (2006), 195-202.

[18] B.Sahin, *CR-warped product submanifolds of nearly Kaehler manifolds,* Contributions to Algebra and Geometry, 49(2) (2008), 383-397.

[19] K.Sekigawa, *Some CR-submanifolds in 6-dimensional sphere,* Tensor (N.S.), 41 (1984), 13 -20.