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SCR-lightlike Warped Product Submanifolds in Indefinite Sasakian Manifolds

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Abstract

In this paper, we study SCR-lightlike warped product submanifolds in an indefinite Sasakian manifold and prove some characterizations of such submanifolds. We also prove that every warped product SCR-lightlike submanifold $N_{\perp} \times_{\lambda} N_{T}$ in an indefinite Sasakian manifold is a SCR-lightlike product. An example in support of the existence of SCR-lightlike warped product of the type $N_{T} \times_{\lambda} N_{\perp}$ is given.

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1. Introduction

It is well known that the most of spacetime models are warped products. These submanifolds were introduced by Bishop and O'Neill[2] and studied by Beem et al[1] who first observed that the well known exact solutions of the Einstein field equations possess warped product structure. In [11], Youm gave an interesting application of warped product in the study of null geodesics and branes. In 2001, Chen ([3],[4]) initiated the study of warped product CR-submanifolds of Kaehler manifolds and obtained many interesting results. Recently, generic warped product submanifolds in nearly Kaehler manifolds were studied by V.A. Khan and K.A. Khan [10]. On the other hand, CR-lightlike warped product submanifolds of an indefinite Kaehler manifold were studied by R. Rani et al[13] and obtained some characterizations of CR-lightlike warped product submanifolds. The notion of screen Cauchy Riemann (SCR) lightlike submanifolds of Kaehler manifolds were introduced and investigated by B. Sahin [7]. Later on, Duggal and Sahin[8] gave a general notion of contact CR and SCR-lightlike submanifolds of an indefinite Sasakian manifold and studied the integrability conditions of their distributions, geometry of leaves of the distributions involved as well as other properties of this submanifold.

In view of physical applications of warped product submanifolds in different areas of mathematics and mathematical physics and no information available on SCR-lightlike warped product submanifolds of indefinite Sasakian manifolds, motivated us to study these submanifolds.

The paper is arranged as follows. In section 2, we recall some results on SCR-lightlike submanifolds, give basic formulae and definition for an indefinite Sasakian manifold and its lightlike submanifolds along with the definition of warped product manifolds which we shall use later. In section 3, we prove some results on the existence of SCR-lightlike warped product submanifolds and establish that every warped product SCR-lightlike submanifold $N_\perp \times_\lambda N_T$ in an indefinite Sasakian manifold is a SCR-lightlike product.

2. Preliminaries

We follow [6] for the notation and fundamental equations for lightlike submanifolds used in this paper. A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \overline{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution $Rad\ TM = TM \cap TM^\perp$ is of rank r, where $1 \leq r \leq m$ and

$$TM^\perp = \cup\{U \in T_x\overline{M} : \overline{g}(U,V) = 0, \forall\, V \in T_x\overline{M}\}.$$

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad\,TM$ in TM, i.e,

$$TM = Rad\,TM \perp S(TM).$$

Consider a screen transversal vector bundle $S(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $Rad\ TM$ in $TM^\perp$. Since for any local basis $\{\xi_i\}$ of $Rad\ TM$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $[S(TM)]^\perp$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [[6],pp-144]. Let $tr(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\overline{M}|_M$.
Then,

$$tr(TM) = ltr(TM) \perp S(TM^\perp),$$

$$T\overline{M} = S(TM) \perp [Rad\,TM \oplus ltr(TM)] \perp S(TM^\perp). \tag{2.1}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to the decomposition (2.1), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y) \ \ \forall X, Y \in \Gamma(TM), \tag{2.2}$$

and

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y; A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively, ∇ and ∇^t are linear connection on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N),$$
$$\overline{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W), \tag{2.4}$$

$\forall X, Y \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. If we denote the projection of TM on $S(TM)$ by P, then by using (2.2), (2.3)-(2.4) and the fact that $\overline{\nabla}$ is a metric connection, we obtain

$$\overline{g}(h^s(X,Y), W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.5}$$

$$\overline{g}(D^s(X,N), W) = \overline{g}(N, A_W X).$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY),$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi,$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\, TM)$. By using above equation we obtain

$$\overline{g}(h^l(X,PY),\xi) = g(A^*_\xi X, PY),$$

$$\overline{g}(h^*(X,PY),N) = g(A_N X, PY),$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, A^*_\xi \xi = 0.$$

An odd dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is called a contact metric manifold [5] if there exists a (1,1) tensor field ϕ, a vector field V, called the characteristic vector field, and its 1-form η satisfying

$$\left.\begin{array}{l} \overline{g}(\phi X, \phi Y) = \overline{g}(X,Y) - \epsilon\eta(X)\eta(Y), \overline{g}(V,V) = \epsilon \\ \phi^2 X = -X + \eta(X)V, \overline{g}(X,V) = \epsilon\eta(X), \\ d\eta(X,Y) = \overline{g}(X, \phi Y), \forall X, Y \in \Gamma(T\overline{M}), \end{array}\right\} \tag{2.6}$$

where $\epsilon = \pm 1$. One can easily verify that $\phi V = 0,\ \eta o \phi = 0,\ \eta(V) = \epsilon$.
Then $(\phi, V, \eta, \overline{g})$ is called a contact metric structure of $\overline{M}$. The semi-Riemannian manifold $\overline{M}$ is said to have a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where

N_ϕ is the Nijenhuis tensor field of ϕ [12]. A normal contact metric manifold is called a Sasakian manifold [[12],[14]] for which we have

$$(\overline{\nabla}_X\phi)Y = -\overline{g}(X,Y)V + \epsilon\eta(Y)X. \tag{2.7}$$

$$\overline{\nabla}_X V = \phi X, \tag{2.8}$$

A lightlike submanifold $(M, g, S(TM), S(TM^\perp))$,tangent to the structure vector field V, of an indefinite Sasakian manifold $\overline{M}$ is said to be a contact SCR-lightlike submanifold [8] if the following conditions are satisfied:

1. There exists real non-null distributions D and $D^\perp$ such that

$$S(TM) = D \oplus D^\perp \perp \{V\}, \phi(D^\perp) \subset S(TM^\perp), D \cap D^\perp = \{0\}$$

where $D^\perp$ is orthogonally complementary to $D\perp\{V\}$ in $S(TM)$.

2. The distribution D and $Rad\,TM$ are invariant w.r.t ϕ.

Since $Rad\ TM$ is invariant, one can easily see that $ltr(TM)$ is also invariant with respect to ϕ. Hence we have

$$TM = \overline{D} \oplus D^\perp \perp \{V\}, \overline{D} = D \perp Rad\,TM.$$

If we denote the projections on $\overline{D}\perp\{V\}$ and $D^\perp$ by $\overline{P}$ and Q respectively, then for any vector field X tangent to M we can write

$$X = \overline{P}X + QX, \tag{2.9}$$

where $\overline{P}X = P_1X + \eta(X)V (P_1$ is a projection on $\overline{D})$ and $QX \in D^\perp$. Applying ϕ to (2.9), we get

$$\phi X = TX + FX$$

where $TX = \phi PX$ and $FX = \phi QX$. On the other hand, for $U \in \Gamma(tr(TM))$

$$\phi U = BU + CU,$$

where BU and CU are the tangential and transversal parts of ϕU, respectively.

For the study of SCR-lightlike submanifolds, we need the following results for later use.

Lemma 2.1. ([8]) Let M be a contact $SCR-$lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then

$$(\nabla_X T)Y = A_{FY}X + Bh^s(X,Y) - g(X,Y)V + \eta(Y)X \tag{2.10}$$

$$(\nabla_X F)Y = Ch^s(X,Y) - h^s(X,TY) \tag{2.11}$$

$$\phi h^l(X,Y) = h^l(X,TY) + D^l(X,FY) \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$.

Theorem 2.2. ([8]) Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then

(i) the distribution $D^\perp$ is integrable if and only if

$$A_{\phi X}Y = A_{\phi X}Y, \ \ \forall X, Y \in \Gamma(D^\perp).$$

(ii) the distribution $\overline{D}\perp\{V\}$ is integrable if and only if

$$h^s(X,TY) = h^s(TX,Y), \ \ \forall X, Y \in \Gamma(\overline{\Delta}).$$

(iii) the distribution $\overline{D}$ is not integrable.

Theorem 2.3. ([8]) Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $\overline{D}\perp\{V\}$ defines a totally geodesic foliation in M if and only if $h^s(X,\phi Y)$ has no component in $\phi D^\perp$ for all $X, Y \in \Gamma(\overline{D}\perp\{V\})$.

Theorem 2.4. ([8]) Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $D^\perp$ defines a totally geodesic foliation in M if and only if $A_{\phi Z}W$ has no component in $\overline{D}$ for all $Z, W \in \Gamma(D^\perp)$.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold $M_1 \times_f M_2 = (M_1 \times M_2, g)$, where

$$g = g_1 + f^2 g_2.$$

The function f is called the warping function of the warped product $M_1 \times_f M_2$. The gradient of f, written as ∇f, is defined by $g(\nabla f, X) = Xf$ for all $X \in \Gamma(TM)$. If X is tangent to M_1 and W is tangent to M_2, then from Lemma 7.3 of [2], one has

$$\nabla_X W = \nabla_W X = \frac{Xf}{f}W. \tag{2.13}$$

It is known that if $M = M_1 \times_f M_2$ is a warped product manifold, then M_1 is totally geodesic and M_2 is totally umbilical[2].

3. SCR-lightlike Warped Product Submanifolds

We observe that a warped product SCR-lightlike submanifold of the type $N_\perp \times_\lambda N_T$ in indefinite Sasakian manifolds is a SCR-lightlike product [6], that is, such warped products don't exist in indefinite Sasakian manifolds. For the existence of warped product $N_T \times_\lambda N_\perp$ in indefinite Sasakian manifolds, we give the following example. We first recall the Sasakian structure defined on R_q^{2m+1}.

Denoting $(R_q^{2m+1}, \phi_0, V, \eta, g)$ as the manifold R_q^{2m+1} with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), V = 2\partial z$$

$$\overline{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i)$$

$$\phi_0(\sum_{i=1}^{m}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) + \sum_{i=1}^{m} Y_i y^i \partial z$$

where (x^i, y^i, z) are the cartesian co-ordinates.

Example 3.1. Let M be a submanifold in the semi-Euclidean space $(R_2^{11}, \overline{g})$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z\}$ and let the submanifold M be defined by the equations

$$x^1 = u, x^2 = s, x^3 = u\cos w, x^4 = u\sin w, x^5 = s$$
$$y^1 = -v, y^2 = -t, y^3 = -v\cos w, y^4 = -v\sin w, y^5 = -t.$$

Then the tangent bundle of M is spanned by

$$\xi_1 = \partial x_1 + \cos w\partial x_3 + \sin w\partial x_4 + (y^1 + y^3\cos w + y^4\sin w)\partial z,$$
$$\xi_2 = -\partial y_1 - \cos w\partial y_3 - \sin w\partial y_4, \quad X_1 = \partial x_2 + \partial x_5 + (y^2 + y^5)\partial z,$$
$$X_2 = -\partial y_2 - \partial y_5,$$
$$Z = -u\sin w\partial x_3 + u\cos w\partial x_4 + v\sin w\partial y_3 - v\cos w\partial y_4 + (-y^3 u\sin w + y^4 u\cos w)\partial z,$$
$$V = \partial z.$$

Hence M is a 2-lightlike submanifold with invariant $Rad\,TM = span\{\xi_1, \xi_2\}$. One can easily verify that the lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}\{-\partial x_1 + \cos w\partial x_3 + \sin w\partial x_4 + (-y^1 + y^3\cos w + y^4\sin w)\partial z\},$$

$$N_2 = \frac{1}{2}(\partial y_1 - \cos w\partial y_3 - \sin w\partial y_4)$$

and the screen transversal bundle $S(TM^\perp)$ is spanned by

$$W = u\sin w\partial y_3 - u\cos w\partial y_4 + v\sin w\partial x_3 - v\cos w\partial x_4 + (y^3 v\sin w - y^4 v\cos w)\partial z.$$

A direct calculation shows that

$$\phi X_1 = X_2, \quad \phi X_2 = -X_1, \quad \phi Z = W$$

and ϕZ is orthogonal to Z. Let $D = span\{X_1, X_2\}$ and $D^\perp = span\{Z\}$. Then D is an invariant distribution, $D^\perp$ is an anti-invariant distribution, that is, $\phi D^\perp \subset (S(TM^\perp))$ and the vector field V is tangent to $\overline{D} = D \perp RadTM$. Hence, M is a proper SCR-lightlike submanifold in R_2^{11}. Moreover, it is easy to see that $\overline{D} \perp \{V\}$

and $D^{\perp}$ are integrable. If we denote the integral manifolds of $\overline{D}\perp\{V\}$ and $D^{\perp}$ by N_T and $N_{\perp}$ respectively, then induced metric tensor of M is

$$\begin{aligned} dr^2 &= 0(du^2+dv^2)+ds^2+dt^2+dv^2+(u^2+v^2)dw^2 \\ &= g_{N_T}+(u^2+v^2)g_{N_{\perp}}, \end{aligned}$$

from which we conclude that M is warped product SCR-lightlike submanifold of R_2^{11} with warping function $f=\sqrt{u^2+v^2}$.

Now, we recall the following definition of SCR-lightlike product submanifolds.

Definition 3.2. ([6]) A SCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is called a SCR- lightlike product if both the distribution $\overline{D}\perp\{V\}$ and $D^{\perp}$ defines totally geodesic foliation in M.

If the leaves of $D^{\perp}$ and $\overline{D}\perp\{V\}$ are denoted by $N_{\perp}$ and N_T respectively, then we have:

Theorem 3.3. Let M be a SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. If $M=N_{\perp}\times_{\lambda}N_T$ is a warped product SCR-lightlike submanifold and V is tangent to N_T, then M is a SCR-lightlike product.

Proof. Suppose that $M=N_{\perp}\times_{\lambda}N_T$ is a warped product SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then N_T is a totally geodesic submanifold of M[2]. Using this fact, for any $Z,W\in\Gamma(N_{\perp})$ and $X\in\Gamma(N_T)$, we obtain

$$g(\nabla_Z W,X)=0.$$

Also, by the use of (2.3),(2.4) and (2.7) in $\overline{\nabla}_Z\phi W=(\overline{\nabla}_Z\phi)W+\phi\overline{\nabla}_Z W$, we get

$$\begin{aligned} -A_{\phi W}Z+\nabla^s_Z\phi W+D^l(Z,\phi W) &= -g(Z,W)V+\eta(W)Z+\phi\nabla_Z W+\phi h^l(Z,W) \\ &\quad +\phi h^s(Z,W). \end{aligned}$$

Taking inner product of the above equation with ϕY, $Y\in\Gamma(N_T)$, and then using (2.6) and (2.12) in the resulting equation, we obtain

$$g(A_{\phi W}Z,\phi Y)=0,$$

which shows that $A_{\phi W}Z$ has no component in $\overline{D}\perp\{V\}$. In view of Theorem 2.4 [8], we conclude that the distribution $D^{\perp}$ defines totally geodesic foliation in M. For the distribution $\overline{D}\perp\{V\}$ to define totally geodesic foliation in M, we denote the second fundamental form and the shape operator of N_T in M by h^T and A^T respectively. Then, for any $X,Y\in\Gamma(N_T)$ and $Z\in\Gamma(N_{\perp})$, from (2.3) and (2.13) we have

$$g(h^T(X,Y),Z)=-(Zln\lambda)g(X,Y)-\overline{g}(h^l(X,Z),Y). \tag{3.1}$$

If we denote the second fundamental form of N_T in $\overline{M}$ by $\hat{h}$, then

$$\hat{h}(X,Y)=h^T(X,Y)+h^l(X,Y)+h^s(X,Y) \tag{3.2}$$

for any $X, Y \in \Gamma(N_T)$. Taking inner product of (3.2) with Z and making use of (3.1), we get

$$\overline{g}(\hat{h}(X,Y),Z) = -(Zln\lambda)g(X,Y) - \overline{g}(h^l(X,Z),Y). \tag{3.3}$$

We note that N_T is a holomorphic submanifold of $\overline{M}$ and for this holomorphic submanifold, one has $\hat{h}(X,\phi Y) = \hat{h}(\phi X, Y) = \phi\hat{h}(X,Y)$. Taking this fact into account, the equation (3.3) can be written as

$$\overline{g}(\hat{h}(X,Y),Z) = (Zln\lambda)g(X,Y) - (Zln\lambda)\eta(X)\eta(Y) + \overline{g}(h^l(\phi X,Z),\phi Y). \tag{3.4}$$

Adding (3.3) and (3.4), we get

$$\overline{g}(\hat{h}(X,Y),Z) = \frac{1}{2}\{-(Zln\lambda)\eta(X)\eta(Y)$$
$$-\overline{g}(h^l(X,Z),Y) + \overline{g}(h^l(\phi X,Z),\phi Y)\Big\}. \tag{3.5}$$

Taking inner product of (3.2) with ϕZ and using (2.3),(2.6),(2.7),(2.13), (3.5), a direct calculation shows that

$$\overline{g}(h^s(X,Y),\phi Z) = 0,$$

from which we conclude that $h^s(X,Y)$ has no component in $\phi(D^\perp)$ for any $X, Y \in \Gamma(\overline{D}\perp\{V\})$. In view of Theorem 2.3 [8], $\overline{D}\perp\{V\}$ defines a totally geodesic foliation in M and hence M is a SCR-lightlike product.

For the study of SCR-lightlike warped product submanifolds, we need the following result.

Lemma 3.4. Let M be a proper SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then $A_{\phi Z}W - A_{\phi W}Z$ has no component in $D^\perp$ for all $Z, W \in \Gamma(D^\perp)$.

Proof. For any $Z, W \in \Gamma(D^\perp)$, from (2.10) we have

$$A_{\phi Z}W = T\nabla_Z W - Bh^s(Z,W) + g(Z,W)V \tag{3.6}$$

Interchanging Z and W in (3.6) and subtracting the resulting equation from (3.6) we obtain

$$A_{\phi Z}W - A_{\phi W}Z = T\nabla_Z W - T\nabla_W Z. \tag{3.7}$$

Taking inner product of (3.7) with $Y \in \Gamma(D^\perp)$, we get

$$g(A_{\phi Z}W - A_{\phi W}Z, Y) = 0,$$

which proves our assertion.

A SCR-lightlike submanifold of an indefinite Sasakian manifold is said to be $D^\perp$-geodesic if $h(U,V) = 0$ for any $U, V \in \Gamma(D^\perp)$.

Using above definition, a characterization of SCR-lightlike warped product submanifolds is given by the following

Theorem 3.5. A proper $D^{\perp}$-geodesic SCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is a SCR-lightlike warped product if and only if there exists a C^{∞}-function μ on M with $U\mu = 0$ for any $U \in \Gamma(D^{\perp})$ and

$$A_{\phi Z}X = \{\eta(X) - (\phi X)\mu\}Z \tag{3.8}$$

for any $X \in \Gamma(\overline{D}\perp\{V\})$ and $Z \in \Gamma(D^{\perp})$.

Proof. Suppose that the proper $D^{\perp}$-geodesic SCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is a SCR-lightlike warped product. Then for any $X \in N_T$ and $Z \in N_{\perp}$, from (2.3),(2.4) and (2.7) together with the fact that $\overline{\nabla}_X\phi Z = (\overline{\nabla}_X\phi)Z + \phi\overline{\nabla}_X Z$ we have

$$-A_{\phi Z}X + \nabla^s_X\phi Z + D^l(X,\phi Z) = \phi(\nabla_X Z + h^l(X,Z) + h^s(X,Z)).$$

Using (2.3), (2.7) and (2.13) in the above equation, we get

$$\begin{aligned} &-A_{\phi Z}X + \nabla^s_X\phi Z + D^l(X,\phi Z) \\ &= (\phi X ln\lambda)Z + h^l(\phi X, Z) + h^s(\phi X, Z) - \eta(X)Z. \end{aligned} \tag{3.9}$$

Hence (3.8) follows from (3.9) by comparing its tangential components and for any $W \in \Gamma(D^{\perp})$, we have $W(ln\lambda) = 0$, where we have used the fact that λ is a function on N_T.

Conversely, assume that (3.8) holds together with $U\mu = 0$, for some C^{∞} function μ on M for all $U \in \Gamma(D^{\perp})$. Then, for $Y \in \Gamma(\overline{D}\perp\{V\})$ and $Z \in \Gamma(D^{\perp})$, we have

$$g(A_{\phi Z}X, \phi Y) = \{\eta(X) - (\phi X)\mu\}g(Z,\phi Y) = 0. \tag{3.10}$$

Making use of (2.5) and (3.10), we obtain

$$\overline{g}(h^s(X,\phi Y),\phi Z) = 0,$$

which shows that $h^s(X,\phi Y)$ has no component in $\phi(D^{\perp})$. Thus the integrability of $\overline{D}\perp\{V\}$ and the totally geodesicness of its integral manifold N^T are followed from Theorem 2.3.[8]. In respect of the integrability of $D^{\perp}$, taking inner product of (3.8) with $W \in \Gamma(D^{\perp})$ we get

$$g(A_{\phi Z}X, W) = (\eta(X) - (\phi X)(\mu))g(Z,W) \tag{3.11}$$

for $X \in \Gamma(\overline{D}\perp\{V\})$ and $Z \in \Gamma(D^{\perp})$. Interchanging Z and W in (3.11) and subtracting the resulting equation from (3.11), we get

$$g(A_{\phi Z}W - A_{\phi W}Z, X) = 0. \tag{3.12}$$

Thus the integrability of $D^{\perp}$ follows from (3.12), Lemma 3.4 and Theorem 2.2 [8]. Now, taking inner product of (3.8) with $W \in \Gamma(D^{\perp})$ and using (2.3),(2.4) and (2.7), a direct calculation shows that

$$-(\phi X)(\mu)g(Z,W) = g(\nabla_Z W, \phi X) + g(h^l(Z,W), \phi X). \tag{3.13}$$

Making use $D^{\perp}$-geodesicness of M in (3.13), we get

$$-(\phi X)(\mu)g(Z,W) = g(\nabla_Z W, \phi X),$$

from which we have

$$g(\nabla_Z W, \phi X) = -g(\nabla\mu, \phi X)g(Z,W). \tag{3.14}$$

If we denote $h^{\perp}$ as the second fundamental form of the leaves $N_{\perp}$ of $D^{\perp}$ and $\nabla^{\perp}$ as the metric connection of $N_{\perp}$ immersed in M, then

$$g(h^{\perp}(Z,W), \phi X) = g(\nabla_Z W, \phi X) \tag{3.15}$$

for any $X \in \Gamma(\overline{D}\perp\{V\})$. By the use of (3.14) and (3.15), we obtain

$$g(h^{\perp}(Z,W), \phi X) = -g(\nabla\mu, \phi X)g(Z,W),$$

which implies

$$h^{\perp}(Z,W) = -\nabla\mu g(Z,W).$$

The above equation shows that the leaves of $D^{\perp}$ are totally umbilical in M. The condition $U\mu = 0$ ensures that the mean curvature vector is parallel. That is, the leaves of $D^{\perp}$ are extrinsic spheres in M. Hence by a result of Hiepko [9] we conclude that M is a SCR-lightlike warped product submanifold $N_T \times_\lambda N_{\perp}$ of $\overline{M}$.

We need the following relations for the study of SCR-lightlike warped product submanifolds.

Lemma 3.6. Let $M = N_T \times_\lambda N_{\perp}$ be SCR-lightlike warped product submanifolds of an indefinite Sasakian manifold $\overline{M}$. Then

$$(\nabla_Z T)X = TX(ln\lambda)Z \tag{3.16}$$

$$(\nabla_U T)Z = T(\nabla ln\lambda)g(Z,U) \tag{3.17}$$

for any $U \in \Gamma(TM), X \in \Gamma(N_T)$ and $Z \in \Gamma(N_{\perp})$.

Proof. From (2.13), for any $X \in \Gamma(N_T)$ and $Z \in \Gamma(N_{\perp})$, we have

$$(\nabla_Z T)X = TX(ln\lambda)Z. \tag{3.18}$$

Moreover, for $U \in \Gamma(TM)$ and $Z \in \Gamma(N_{\perp})$ we observe that

$$(\nabla_U T)Z = -T\nabla_U Z,$$

which shows that $(\nabla_U T)Z \in \Gamma(N_T)$. Using this condition together with (2.3) and (2.13), for any $X \in \Gamma(D)$, we obtain

$$g((\nabla_U T)Z, X) = -(TX ln\lambda)g(Z, U). \tag{3.19}$$

Thus our assertion follows from (3.19) together with the definition of gradient of λ and the non-degeneracy of D.

Some more characterizations of SCR-lightlike warped product submanifolds are given by the following theorems.

Theorem 3.7. A proper $D^{\perp}$-geodesic SCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is a SCR-lightlike warped product submanifold if and only if there exists a C^{∞}-function μ on M with $Z\mu = 0$ for any $Z \in \Gamma(D^{\perp})$ and

$$(\nabla_U T)W = (TW(\mu))QU + g(QU, QW)\phi(\nabla\mu), \tag{3.20}$$

for $U, W \in \Gamma(TM)$.

Proof. Assume that $M = N_T \times_\lambda N_\perp$ is a SCR-lightlike warped product submanifold of an indefinite Sasakian manifold $\overline{M}$. For any $U, W \in \Gamma(TM)$, from (2.9) we have

$$(\nabla_U T)W = (\nabla_{\overline{P}U} T)\overline{P}W + (\nabla_{QU} T)\overline{P}W + (\nabla_U T)QW. \tag{3.21}$$

By the use of (2.10) and the fact that $\overline{D}\perp\{V\}$ defines totally geodesic foliation in M, we arrive at

$$(\nabla_{\overline{P}U} T)\overline{P}W = 0. \tag{3.22}$$

On the other hand, from (3.16) and (3.17) we have

$$(\nabla_{QU} T)\overline{P}W = TW(ln\lambda)QU \tag{3.23}$$

and

$$(\nabla_U T)QW = g(QU, QW)T(\nabla ln\lambda). \tag{3.24}$$

Hence our assertion follows from (3.21) - (3.24).

Conversely, suppose that (3.20) holds together with $Z\mu = 0$ for some C^{∞}-function μ on M and $Z \in \Gamma(D^{\perp})$. Then for any $U, W \in \Gamma(\overline{D}\perp\{V\})$, using(3.20), we obtain $(\nabla_U T)W = 0$, from which we have

$$Bh^s(U, W) = g(U, W)V + \eta(W)U, \tag{3.25}$$

where we have used (2.10). Taking inner product of (3.25) with $Z \in D^{\perp}$, we get

$$g(h^s(U, W), \phi Z) = 0,$$

which shows that the distribution $\overline{D}\perp\{V\}$ is integrable and the leaves of $\overline{D}\perp\{V\}$ are totally geodesic in M(Theorem 2.3 [8]). For the integrability of $D^{\perp}$, taking the inner product of (3.20) with $Z \in \Gamma(TM)$ and using (2.10), we arrive at

$$g(A_{FW}U + Bh^s(U, W) - g(U, W)V, Z) = g(U, W)g(\phi(\nabla_\mu), Z) \tag{3.26}$$

Interchanging U and W in (3.26) and subtracting the resulting equation from (3.26), we get

$$g(A_{FW}U - A_{FU}W, Z) = 0. \tag{3.27}$$

Hence the integrability of $D^{\perp}$ follows from (3.27) and Theorem 2.5 [8].
Also, from (3.20) we have

$$(\nabla_U T)W = g(QU, QW)\phi(\nabla\mu) \tag{3.28}$$

for any $U, W \in \Gamma(D^{\perp})$. Taking inner product of (3.28) with $X \in \Gamma(\overline{D})$, we get

$$g((\nabla_U T)W, X) = g(QU, QW)g(\phi(\nabla\mu), X). \tag{3.29}$$

On the other hand, from (2.3), (2.4), (2.7) and (2.10) we obtain

$$g((\nabla_U T)W, X) = g(\nabla_U W, \phi X) + g(h^l(U, W), \phi X). \tag{3.30}$$

Using $D^{\perp}$-geodesicness of M in (3.30), we obtain

$$g((\nabla_U T)W, X) = g(\nabla_U W, \phi X) \tag{3.31}$$

From (3.29) and (3.31), we have

$$g(\nabla_U W, \phi X) = -g(\nabla\mu, \phi X)g(U, W). \tag{3.32}$$

If we denote $h^{\perp}$ and $\nabla^{\perp}$ as the second fundamental form and the metric connection of the leaves of $D^{\perp}$ in M, then

$$g(h^{\perp}(U, W), \phi X) = g(\nabla_U W, \phi X). \tag{3.33}$$

Making use of (3.32) and (3.33), we obtain

$$g(h^{\perp}(U, W), \phi X) = -g(\nabla\mu, \phi X)g(U, W),$$

from which we have

$$h^{\perp}(U, W) = -\nabla\mu g(U, W), \tag{3.34}$$

where we have used the fact that the distribution $\overline{D}$ is non-degenerate. From the equation (3.34), we observe the leaves of the distribution $D^{\perp}$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover, from $Z\mu = 0$ for any $Z \in \Gamma(D^{\perp})$, we conclude that the mean curvature vector is parallel. Thus, the leaves of $D^{\perp}$ are extrinsic spheres in M. Hence by a result of Hiepko [9], M is a SCR-lightlike warped product submanifold of $\overline{M}$.

Theorem 3.8. Let M be a $D^{\perp}$-geodesic SCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then M is a lightlike warped product if and only if there exists a C^{∞}-function μ on M with $Z\mu = 0$ for any $Z \in \Gamma(D^{\perp})$ and

$$\overline{g}((\nabla_U F)W, \phi Z) = -\overline{P}W(\mu)g(U, Z) \tag{3.35}$$

for all $U, W \in \Gamma(TM)$.

Proof. If M is a SCR-lightlike warped product submanifold of indefinite Sasakian manifold $\overline{M}$, then $\overline{D}\bot\{V\}$ defines totally geodesic foliation in M. For $U, W \in \Gamma(\overline{D}\bot\{V\})$, from (2.11)we have

$$\overline{g}((\nabla_U F)W, \phi Z) = 0.$$

Similarly, using (2.11), for any $U \in \Gamma(D^\perp), V \in \Gamma(D^\perp)$ or $U, V \in \Gamma(D^\perp)$ we get

$$\overline{g}((\nabla_U F)W, \phi Z) = 0. \tag{3.36}$$

If we take $U \in \Gamma(D^\perp)$ and $W \in \Gamma(\overline{D}\bot\{V\})$, then from (2.13) we obtain

$$\overline{g}((\nabla_U F)W, \phi Z) = -(W ln\lambda)g(U, Z). \tag{3.37}$$

Hence our assertion follows from (3.36) and (3.37).

Conversely, suppose that there exists a C^∞-function μ on M with $Z\mu = 0$ for any $Z \in \Gamma(D^\perp)$ and (3.35) holds. Then for any $X, Y \in \Gamma(\overline{D}\bot\{V\})$ and $W \in \Gamma(D^\perp)$, one can easily verify that

$$g((\nabla_X F)Y, \phi Z) = 0,$$

from which we have

$$g(h^s(X, TY), \phi Z) = 0, \tag{3.38}$$

where we have used (2.11). From equation (3.38)and Theorem 2.3 [8], we conclude that the distribution $\overline{D}\bot\{V\}$ is integrable and its leaves are totally geodesic in M. For the integrability of $D^\perp$, from (2.11) and the $D^\perp$-geodesicness of M we have

$$\nabla^s_U FW - F\nabla_U W = 0. \tag{3.39}$$

Interchanging U and W in (3.39)and subtracting the resulting equation from (3.39), we get

$$\nabla^s_U FW - \nabla^s_W FU = F[U, W]$$

which shows that $D^\perp$ is integrable.

Now, from (3.35) for any $Z \in \Gamma(D^\perp)$, we obtain

$$g((\nabla_W F)X, \phi Z) = -X(\mu)g(Z, W),$$

from which we have

$$\overline{g}(h^s(\phi X, W), \phi Z) = X(\mu)g(Z, W),$$

where we have used (2.11). Replacing X by ϕX in above equation and using (2.8), we arrive at

$$-\overline{g}(h^s(X, W), \phi Z) + \eta(X)g(Z, W) = \phi X(\mu)g(Z, W). \tag{3.40}$$

Denoting the leaves of $D^{\perp}$ by $N_{\perp}$, the second fundamental form of $D^{\perp}$ in M by $h^{\perp}$ and the metric connection of $D^{\perp}$ in M by $\nabla^{\perp}$. Then

$$g(h^{\perp}(Z,W),\phi X) = g(\nabla_Z W, \phi X). \tag{3.41}$$

On the other hand, from (2.3) and (2.7) we have

$$\overline{g}(h^s(X,W),\phi Z) = \overline{g}(\phi X, \overline{\nabla}_W Z) + \eta(X)g(Z,W).$$

Using (2.3) and the $D^{\perp}$-geodesicness of M in the above equation, we get

$$g(\nabla_W Z, \phi X) = -\overline{g}(h^s(X,W),\phi Z) + \eta(X)g(Z,W),$$

from which we have

$$g(\nabla_W Z, \phi X) = g(\nabla\mu, \phi X)g(Z,W), \tag{3.42}$$

where we have used (3.40). From (3.41) and (3.42), we have

$$h^{\perp}(Z,W) = \nabla\mu g(Z,W),$$

which shows that $N_{\perp}$ is totally umbilical in M with mean curvature vector $\nabla\mu$. Also, from the condition $Z(\mu) = 0$ for any $Z \in \Gamma(D^{\perp})$, we infer that the mean curvature vector is parallel. Consequently, the leaves $N_{\perp}$ of $D^{\perp}$ are extrinsic spheres in M. Hence by a result of Hiepko [9], M is a SCR-lightlike warped product submanifold of $\overline{M}$.

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