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New Sequences which have Limit π

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Abstract
We consider a regular polygon with even number of sides, denoted by RPES for brevity and in analogy with the relations a circle has with its diameter with regard to π, we consider perimeter P_{2n} and area A_{2n} of RPES of sides $2n$ and its diagonal d_{2n} and set

$$S_{2n} = \frac{P_{2n}}{d_{2n}} \quad \text{and} \quad S_{2n(a)} = \frac{A_{2n}}{(d_{2n/2})^2} \quad n = 2, 3, 4, \ldots$$

and show that each of these two sequences has limit π. It is also shown that a sequence obtained from a regular polygon inscribed in a circle of radius 1, is a special case of S_{2n} and that another sequence obtained from the superscribed regular polygon on the same circle and claimed to have limit π does not, indeed, have limit π. The condition necessary for its convergence to π is indicated. The polygons that we consider are not required to be either inscribed in or superscribed on a fixed circle.

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1. Introduction

The history of theoretical determination of the value of π dates back to Archimedes of Syracuse (287 - 212 B. C). Extending his idea, using trigonometry and algebra O'Connor and Robertson [1] have given sequences a_n and b_n where

a_n = Perimeter of a regular polygon of sides K
and superscribed on a circle of radius 1/diameter of the circle. (1.1)

b_n = Perimeter of a regular polygon of sides$K$
and inscribed in a circle of radius 1/diameter of the circle. (1.2)

It is shown that

$$a_n = K \tan \frac{\pi}{K}, b_n = K \sin \frac{\pi}{K} \tag{1.3}$$

where $K = 3.2^{n-1}$.
It is claimed that a_n and b_n both have limit π. But as is shown later here that a_n does not have limit π and b_n is a special case of S_{2n}.

2. Expressions for S_{2n} and $S_{2n(a)}$

With the meaning given to P_{2n} and d_{2n} as above we claim that

$$S_{2n} = 2n \sin \frac{\pi}{2n}, \qquad n = 2, 3, 4, \ldots \tag{2.1}$$

We indicate the proof by giving an illustration. It will show how we arrive at the general formula (2.1).
For $n = 4$, RPES is an octagon. Let each side of RPES be of length a. Since we are interested in finding the length of the diagonal, it is sufficient to look at half the figure of the octagon. This half figure has four isosceles triangles and each triangle subtends angle of 45 degrees at the centre O. Draw OM perpendicular to AB. Let half diagonal OA be denoted by x. Then $x.\sin\frac{\pi}{8} = \frac{a}{2}$ and

$AM = \frac{a}{2}$ so that $S_8 = P_8/d_8 = 8a/2x = 8\sin\frac{\pi}{8}$
Formula (2.1) gives the same result for $n = 4$.
For $n = 2$, RPES is a square. It is easy to see that $S_4 = 2\sqrt{2} = 4\sin\frac{\pi}{4}$.
For $n = 3$, RPES is a hexagon which has six equilateral triangles. Its diagonal is therefore $2a$ where a is the length of each side of the hexagon. So $S_6 = \frac{6a}{2a} = 3 = 6\sin\frac{\pi}{6}$. Thus the sequence $S_4, S_6, S_8, S_{10}, S_{12}, S_{16}, \cdots, S_{90}, \cdots, S_{180}$ is 2.828, 3, 3.061, 3.09, 3.110, 3.21, $\cdots$, 3.1401, $\cdots$ 3.1410, with limit $\pi = 3.14159$.
It is easy to see that the sequence b_n given in (1.3) is a special case of S_{2n} given in (2.1).
Now let A_{2n} be the area of RPES of sides $2n$ and, as before, d_{2n} denote its diagonal. It is known that for a circle of radius a, area of the circle / $a^2 = \pi$. In analogy with this we set

$$S_{2n(a)} = \frac{A_{2n}}{(d_{2n/2})^2}$$

We claim that

$$S_{2n(a)} = n\sin\frac{\pi}{n}, \quad n = 2, 3, 4, \cdots \tag{2.2}$$

To illustrate the method by which this formula is obtained, consider Fig. 2.1.

Here area of $\Delta AOB = \frac{a}{2}x\cos\frac{\pi}{8}$, $d_{2n/2} = x = \dfrac{a}{2}\dfrac{1}{\sin\frac{\pi}{8}}$. Since the octagon has 8 isosceles triangles of equal area

$$S_{8(a)} = 4ax\cos\frac{\pi}{8}/x^2 = 4\sin\frac{\pi}{4}$$

For n = 4, formula (2.2) gives the same result.

For a square of side a , area $= a^2$ and $diagonal/2 = a/\sqrt{2}$. And so

$$S_{4(a)} = 2 = 2\sin\frac{\pi}{2} \tag{2.3}$$

For a hexagon of side a , each of its six equilateral triangles has area $= \frac{a^2}{2}\sin\frac{\pi}{3}$, $\frac{d_6}{2} = a$. Hence

$$S_{6(a)} = \frac{3a^2\sin\pi/3}{a^2} = 3\sin\frac{\pi}{3} \tag{2.4}$$

Formula (2.2) gives the same results for n=2, and n=3. Thus the sequence $S_{4(a)}, S_{6(a)}, S_{8(a)}, S_{10(a)}, S_{12(a)}, \cdots S_{90(a)}, \cdots .S_{180(a)}$, is 2, 2.598 , 2.828, 2.9390, 3, $\cdots$, 3.140, $\cdots$, 3.141, with limit π.

3. Remark

Now refer to Figure 3.1 which contain a regular hexagon inscribed in a circle of radius 1 and another hexagon superscribed on the same circle. This is used in [1] for arriving at the formula for a_n and b_n.

In view of (1.3) and $K = 3.2^{n-1}, a_2, a_3, a_4, \cdots, b_1, b_2, b_3\cdots$ correspond to regular hexagon, 12 sided regular polygon, 24 sided regular polygon. The case of a_2, b_2 is illustrated in Fig 3.1. Again from (1.3) we have

$$a_2 = 6\tan\frac{\pi}{6} \quad , b_2 = 6\sin\frac{\pi}{6} \tag{2.5}$$

Here b_2 has the same expression as that of S_6. We conclude that b_n has the same form as that of S_{2n} and has, therefore, limit π.

It is claimed in [1] that a_n is a decreasing sequence and has limit π. But it is, in fact, not so, for instance

$a_2 = 6\tan\frac{\pi}{6} = 2.1840$, $a_3 = 12\tan\frac{\pi}{12} = 2.5512$, $a_4 = 24\tan\frac{\pi}{24} = 3.1604$, $a_5 = 48\tan\frac{\pi}{48} = 3.2256, \cdots$,

This shows that a_n cannot have its limit as π.

This discrepancy can be avoided if instead of defining a_n as in (1.1), we set a_n = Perimeter of a regular polygon of sides K / its diagonal, then we can arrive at a sequence which has limit π. To see this, according to the new definition of a_n , a_2 is

$$a_2 = 12AT/2OT = 6\tan\frac{\pi}{6}\cos\frac{\pi}{6} = 6\sin\frac{\pi}{6}.$$

So a_2 has the same form as $S_{6(a)}$, which leads to the conclusion that a_n defined in the manner indicated above has limit π.

4. Conclusion

(a) It is important to note that as a diameter divides the perimeter and the area of a circle into two equal parts, so does a diagonal of a regular polygon of even number of sides. It is the diagonal which has to be given the role of the diameter of a circle while dealing with polygons. This leads to accurate results. (b) It is clear from the formulas for S_{2n}, and $S_{2n(a)}$ etc. that each such formula is independent of the length of a side of a polygon and also of the length of its diagonal. The only thing that figures there, that is really of central importance, is the angle subtended by a side of a regular polygon at its centre. So it does not really matter whether such polygons are inscribed or superscribed on a circle.

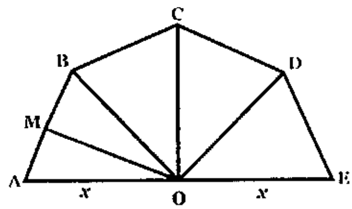

Fig.2.1

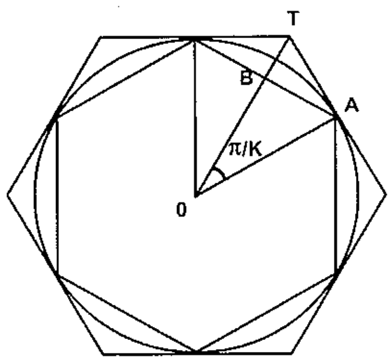

Fig.3.1

$$OA = 1, AB = \sin\tfrac{\pi}{6},\ AT = \tan\tfrac{\pi}{6},\ OT = \tfrac{1}{\cos\frac{\pi}{n}}$$

Reference

O'Connor, J.J. and Robertson, E.F. : *Mac Tutor of History of Mathematics*, (March 14, 2011),) http://www.lewrockwell.com/spl3/history-of-pi.html