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Conformal Correspondence of Finsler Spaces with Special $(\alpha, \beta)-$Metric

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(*Dedicated to Prof. K. S. Amur on his 80th birth year*)

Abstract

The purpose of the present paper is to discuss the conformal transformation of a Finsler space with a special (α, β) metric given by $L^2 = c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2$, where c_1, c_2, c_3 are constants, α is Riemannian metric and β is one form. We have proved that for such a Finsler metric, the Berwald spaces, the locally Minkowski spaces and the projectively flat spaces are not invariant under non-homothetic conformal transformation.

Keywords and Phrases : Randers space, Kropina space, conformal change of metric, generalized Randers space, locally Minkowski space.
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1. Introduction

Let M^n be an n-dimensional differentiable manifold and TM^n be its tangent bundle. The manifold M^n is covered by a coordinate neighbourhood $\{U\}$; in each U of which we have a local coordinate system (x^i). A tangent vector at a point $x = (x^i)$ of U is written as $y^i\big(\frac{\partial}{\partial x^i}\big)_x$ and we have a local co-ordinate system (x^i, y^i) of TM^n over the U.

The manifold M^n equipped with a fundamental function $L(x, y)$ is called a Finsler space $F^n = (M^n, L)$, if L is defined for any point of $TM^n - \{0\}$ and is positively homogeneous of degree one in y^i. If L is positively homogeneous of degree one in α and β, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i\,(x)y^i$ is a one-form on M^n, then F^n is called a Finsler space with $(\alpha, \beta)-$metric ([2], [3]).

Various special $(\alpha, \beta)-$metrics have been considered in the literature. For instance if $L = \alpha+\beta$, then this metric is called Randers metric and corresponding Finsler space is called Randers space [4]. If $L = \frac{\alpha^2}{\beta}$, then this metric is called Kropina metric and corresponding Finsler space is called Kropina space [8].

A Finsler space is called C-reducible Finsler space if its Cartan's C-tensor $C_{ijk} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i \partial y^j \partial y^k}$ is written in the form

$$C_{ijk} = h_{ij}A_k + h_{jk}A_i + h_{ki}A_j, \tag{1.1}$$

where h_{ij} is the angular metric tensor given by $h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j}$ and A_i is a vector field given by $A_i = \frac{1}{n+1}C_{ijk}\, g^{jk}$. In [5], M. Matsumoto and S. Hojo have obtained that a Finsler space of dimension $n \geq 3$, is C-reducible if and only if the F^n is either Randers space or Kropina space.

A Finsler space is called generalized C-reducible [7] if there exists a covariant tensor field K_{ij} of order 2 and a covariant vector field B_i such that

$$C_{ijk} = K_{ij}B_k + K_{jk}B_i + K_{ki}B_j. \tag{1.2}$$

A Finsler space with $(\alpha, \beta)-$metric is generalized C-reducible if and only if $\frac{1}{2}\frac{\partial^3 L^2(\alpha,\beta)}{\partial \beta^3} = 0$. Owing to this fact Park and Choi [6] introduced an (α, β) metric L given by

$$L^2 = c_1\alpha^2 + 2c_2\,\alpha\beta + c_3\beta^2, \tag{1.3}$$

where c_1, c_2, c_3 are constants. In case of $c_1 = c_2 = c_3 = 1$, the metric $L(\alpha, \beta)$ given by (1.3) is a Randers metric, so that the metric $L(\alpha, \beta)$ given by (1.3) may be considered as a generalization of Randers metric. If $c_2 = 0$, the metric $L(\alpha, \beta)$ given by (1.3) is a Riemannian metric and if $c_1c_3 - c_2^2 = 0$, then the metric is Randers metric. Therefore we shall assume that $c_2 \neq 0$ and $c_1c_3 - c_2^2 \neq 0$. A Finsler space with the metric (1.3) is said to be a generalized Randers space.

A Finsler space F^n is called a Berwald space if Berwald's connection coefficient of F^n is linear and F^n is called locally Minkowski if there exist a coordinate system in which L is independent of coordinates (x^i). The Finsler space F^n is called projectively flat if F^n is conformal to a locally Minkowski space.

Let $\nabla_k b_i$ denote covariant derivative of b_i with respect to associated Riemannian connection and R^i_{hjk} be the curvature tensor of Riemannian space (M^n, α), then we have the following theorems [6]:

Theorem 1.1. If $c_1 \neq 0$, $c_3 \neq 0$, then F^n is a Berwald space if and only if $\nabla_k b_i = 0$.

Theorem 1.2. If $c_1 = c_3 = 0$, then F^n is a Berwald space if and only if there exists a covariant vector field (v_k) such that

$$\nabla_k b_i = \frac{1}{2}(v^r b_r a_{ki} - v_i b_k - 2v_k b_i),$$

where $v^r = a^{ri} v_i$.

Theorem 1.3. The Finsler space with an $(\alpha,\beta)-$metric given by (1.3), where $c_1 \neq 0$, $c_3 \neq 0$, is locally Minkowski space if and only if $R^i_{hjk} = 0$ and $\nabla_k b_i = 0$ are satisfied.

Theorem 1.4. A Finsler space with an $(\alpha,\beta)-$metric given by (1.3), is projectively flat if and only if associated Riemannian space (M^n,α) is projectively flat and $\nabla_k b_i = 0$.

2. Conformal Change of Finsler Space

Let two distinct Finsler metric functions $L(x,y)$ and $L^*(x,y)$ be defined over an n-dimensional differentiable manifold M^n, then the two Finsler spaces (M^n,L) and (M^n,L^*) are said to be in conformal correspondence if there exist a function $\sigma(x)$ such that

$$L^* = e^\sigma L. \tag{2.1}$$

If we denote the quantities corresponding to (M^n,L^*) by putting '*' on the superscript position of that quantity, then

$$\begin{aligned} l^{*i} &= \frac{y^i}{L^*} = e^{-\sigma} l^i, \quad l^*_i = \frac{\partial L^*}{\partial y^i} = e^\sigma l_i, \quad {g^*}_{ij} = \frac{1}{2}\frac{\partial^2 L^{*2}}{\partial y^i \partial y^j} = e^{2\sigma} g_{ij}, \\ g^{*ij} &= e^{-2\sigma} g^{ij}, \qquad C^*_{ijk} = e^{2\sigma} C_{ijk}, \qquad C^{*i}_{jk} = g^{*ir} {C^*}_{rjk} = C^i_{jk}. \end{aligned} \tag{2.2}$$

If σ is constant, the conformal transformation is called homothetic. For homothetic transformation $\sigma_i = \frac{\partial \sigma}{\partial x^i} = 0$.

3. Conformal Change of $(\alpha,\beta)-$metric

A conformal change $L \to L^* = e^\sigma L(\alpha,\beta)$ of $(\alpha,\beta)-$metric L is expressed as $(\alpha,\beta) \to (e^\sigma \alpha, e^\sigma \beta)$. Thus a conformal change of $(\alpha,\beta)-$metric is expressed as $(\alpha,\beta) \to (\alpha^*,\beta^*)$, where

$$\alpha^* = e^\sigma\, \alpha, \qquad \text{and} \qquad \beta^* = e^\sigma \beta. \tag{3.1}$$

Therefore, we have

$$a^*_{ij} = e^{2\sigma} a_{ij}, \qquad b^*_i = e^\sigma b_i, \tag{3.2}$$

where $\alpha^* = \sqrt{{a^*}_{ij}(x) y^i y^j}$ and $\beta^* = b^*_i(x) y^i$.

Also we have $b^2 = a^{ij} b_i b_j = a^{*ij} b^*_i b^*_j = b^{*2}$.

Lemma 3.1. In a Finsler space with $(\alpha, \beta)-$metric, the length b of b_i with respect to Riemannian α, is invariant under any conformal change of metric.

As to the metric (1.3) of generalized Randers space we have

$$L^{*2} = e^{2\sigma} L^2 = e^{2\sigma}(c_1\alpha^2 + 2c_2\,\alpha\beta + c_3\beta^2) = c_1\alpha^{*2} + 2c_2\alpha^*\beta^* + c_3\beta^{*2}.$$

Therefore we have the following:

Theorem 3.1. The conformal transformation of a generalized Randers space is a generalized Randers space with same constant coefficients.

If γ^i_{jk} denotes the Christoffel symbol of Riemannian space (M^n, α), then the corresponding quantity of (M^n, α^*) is obtained from (3.2) and is given by

$$\gamma^{*i}_{jk} = \gamma^i_{jk} + (\sigma_k \delta^i_j + \sigma_j \delta^i_k - \sigma^i a_{jk}), \tag{3.3}$$

where $\sigma^i = a^{ij}\sigma_j$, $\sigma_j = \frac{\partial \sigma}{\partial x^j}$.

Thus $\nabla^*_k\, b^*_i = \frac{\partial b^*_i}{\partial x^k} - b^*_h\, \gamma^{*h}_{ik}$ is transformed to

$$\nabla^*_k b^*_i = e^{\sigma}(\nabla_k b_i + b_r \sigma^r a_{ik} - \sigma_i b_k). \tag{3.4}$$

In view of theorem (1.1) and equations (1.3), (3.4) it follows that a generalized Randers-Berwald space is conformally transformed to a generalized Randers-Berwald space if and only if

$$b_r\, \sigma^r a_{ij} = \sigma_i b_j. \tag{3.5}$$

Contracting (3.5) with a^{ij}, we get $(n-1)\, b_r\sigma^r = 0$ which implies that $b_r\sigma^r = 0$ for $n > 1$. Hence from (3.5) we get $\sigma_i b_j = 0$. Since $b_i \neq 0$ we have $\sigma_i = 0$, i.e. the transformation is homothetic. Thus

Theorem 3.2. For $c_1 \neq 0$, $c_3 \neq 0$, a generalized Randers-Berwald space is conformally transformed to a generalized Randers-Berwald space if and only if the transformation is homothetic.

Let F^n with metric (1.3), where $c_1 = c_3 = 0$, be a Berwald space. Then there exists a covariant vector field $v_i(x)$ such that [6]

$$\text{(a)} \quad a_{ij|k} = v_k a_{ij}, \qquad \text{(b)} \quad b_{i|k} = -\frac{1}{2} v_k b_i, \tag{3.6}$$

where $_{|k}$ denote the covariant derivative with respect to Berwald connection $(G^i_{jk}, G^i_j, 0)$. From (3.6)(a), we get

$$G^i_{jk} = \gamma^i_{jk} - \frac{1}{2}(v_k\delta^i_j + v_j\delta^i_k - v^i a_{jk}), \tag{3.7}$$

where $v^i = a^{ij}v_j$. From (3.6)(b) and (3.7), we obtain

$$\nabla_k b_i = \frac{1}{2}(v^r b_r a_{ki} - v_i b_k - 2v_k b_i). \tag{3.8}$$

Now let us suppose that the conformally transformed generalized Randers space F^{*n} is also a Berwald space. Then there exist covariant vector field v^*_i such that

$$\nabla^*_k b^*_i = \frac{1}{2}(v^{*r} b^*_r a^*_{ki} - v^*_i b^*_k - 2v^*_k b^*_i), \tag{3.9}$$

where $v^{*i} = a^{*ij}v^*_j$. From (3.2) we have

$$v^{*r} b^*_r a^*_{ki} = e^{\sigma} v^*_r b^r a_{ki}. \tag{3.10}$$

From (3.4), (3.8) and (3.10), we get

$$v_r b^r a_{ki} - v_i b_k - 2v_k b_i = v^*_r b^r a_{ki} - v^*_i b_k - 2v^*_k b_i. \tag{3.11}$$

Contracting (3.11) with a^{ki} we get $v_r b^r = v^*_r b^r$ and putting this value in (3.11), we have $(v^*_i - v_i)b_k + 2\,(v^*_k - v_k)b_i = 0$.

Contracting this equation by b^k, and using the fact that $v_r b^r = v^*_r b^r$, we get

$$v^*_i = v_i. \tag{3.12}$$

Proposition 3.1. If the generalized Randers- Berwald space for $c_1 = c_3 = 0$, is conformally transformed to the generalized Randers-Berwald space, then the vector field v_i occurring in theorem (1.2) is conformally invariant.

Now we consider the generalized Randers space F^n for $c_1 = c_3 = 0$ and discuss the case that the Berwald space F^n is conformally transformed to a Berwald space or not. In this case from theorem (1.2), F^n is Berwald space if and only if there exists a vector field v_i such that (3.8) holds.

Now from (3.2), (3.8) and (3.12) we get

$$\nabla^*_k b^*_i - \frac{1}{2}(v^{*r} b^*_r a^*_{ki} - v^*_i b^*_k - 2v^*_k b^*_i) = e^{\sigma}(b_r\sigma^r a_{ki} - \sigma_i b_k).$$

Therefore by theorems (1.2) and (3.1), F^{*n} is also a Berwald space if and only if $b_r\sigma^r a_{ki} - \sigma_i b_k = 0$, which gives $\sigma_i = 0$, i.e. the conformal transformation is homothetic.

Theorem 3.3. For $c_1 = c_3 = 0$, the generalized Randers-Berwald space is conformally transformed to the generalized Randers-Berwald space if and only if the transformation is homothetic.

Now let us suppose that the generalized Randers space F^n with $c_1 \neq 0$, $c_3 \neq 0$, is locally Minkowski space. Then in view of theorem (1.3) we have $R^i_{hjk} = 0$ and $\nabla_k b_i = 0$. For conformally transformed generalized Randers space F^*_n, from (3.4) we have

$$\nabla^*_k b^*_i = e^{\sigma}(b_r \sigma^r a_{ik} - \sigma_i b_k) \tag{3.13}$$

and [1]

$$R^{*i}_{hjk} = R^i_{hjk} + \delta^i_k \sigma_{hj} - \delta^i_j \sigma_{hk} + a^{ir}(a_{hj}\sigma_{rk} - a_{hk}\sigma_{rj}) + (\delta^i_k a_{hj} - \delta^i_j a_{hk})\Delta_1 \sigma, \tag{3.14}$$

where

$$\sigma_{ij} = \nabla_i \sigma_j - \sigma_i \sigma_j, \qquad \Delta_1 \sigma = a^{ij}\sigma_i \sigma_j. \tag{3.15}$$

From (3.13) and (3.14) it follows that F^{*n} is also locally Minkowski if and only if $\sigma_i = 0$ i.e. the conformal transformation is homothetic and in this case (3.14) gives $R^{*i}_{hjk} = 0$. Hence we have the following:

Theorem 3.4. If a generalized Randers space F^n with $c_1 \neq 0$, $c_3 \neq 0$, is locally Minkowski, its conformally transformed space F^{*n} is also locally Minkowaski if and only if the transformation is homothetic.

If F^n is projectively flat then from theorem (1.4) we have $\nabla_k b_i = 0$ and associated Riemannian space is projectively flat.

The associated Riemannian space is projectively flat if and only if $\gamma^i_{00} = v_0 y^i$ [3], where '0' denote the contraction with y^i.

From (3.3) and (3.12) it follows that

$$\gamma^{*i}_{00} - v^*_0 y^i = (\gamma^i_{00} - v_0 y^i) + 2\sigma_0 y^i - \alpha^2 \sigma^i. \tag{3.16}$$

Thus if F^n is projectively flat then F^{*n} is also projectively flat if and only if $\sigma_i = 0$ i.e. the conformal transformation is homothetic.

Theorem 3.5. If a generalized Randers space F^n is projectively flat then its conformally transformed space F^{*n} is also projectively flat if and only if the transformation is homothetic.

Conclusion. For a non-homothetic conformal transformation of generalized Randers space

(i) a Berwald space is not conformal to a Berwald space,

(ii) a locally Minkowaski space is not conformal to a locally Minkowaski space,

(iii) a projectively flat space is not conformal to a projectively flat space.

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