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Invariant Objects under a Lorentz Transformation

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1. Introduction

Minkowski space (M, η) is a flat 4-dimensional manifold with Lorentz metric η. Let x^α be coordinates in a Lorentzian coordinate frame $\sum$ on M and $x^{\alpha'}$ be coordinates in another frame $\sum'$ on M which is moving relative to $\sum$ with a uniform velocity in an arbitrary direction. The properties of transformation matrix of $x^{\alpha'}$ into x^α are given in the books of Misner et al [5] and Weinberg [6]. But it appears that these have been rarely used. We use these properties of transformation matrix to derive some significant results. In section 1, we collect all the relevant properties of the transformation matrix and use them in section 2 to establish the invariance of volume element dV and that of differential operator $*dX$ with respect to $\sum$ and $\sum'$, where X is the position vector of a point in $\sum$. In section 3 we apply the results obtained in section 2 to express gradient of a function, divergence and curl of a vectorfield defined on M in an invariant manner. Expressions for these are also obtained in spherical polar coordinates. In section 4, we apply the results of section 3 to study electromagnetism. In addition some properties of hyper-surfaces in (M, η) are also given.

2. Preliminaries

Minkowski space is a globally hyperbolic Lorentzian Manifold M with a flat metric η. Consider a coordinate system (x^0, x^1, x^2, x^3) in the underlying manifold R^4 where $x^0 = ct$. We assume c, the velocity of light is unity so that $x^0 = t$ has the unit of length. x^1, x^2, x^3 are space coordinates as in Euclidian R^3. The basis vectors $\frac{\partial}{\partial x^\alpha}$, $= 0, 1, 2, 3$ denoted by e_α satisfy

$$\begin{aligned} \eta(e_\alpha, e_\beta) &= \eta_{\alpha\beta} = -1 \;\; if \;\; \alpha = \beta = 0 \\ &= 1 \;\; if \;\; \alpha = \beta = 1, 2, 3 \\ &= 0 \;\; if \;\; \alpha \neq \beta \end{aligned} \tag{1.1}$$

A Lorentz transformation is a transformation from one system of coordinates x^α in a Lorentz frame $\sum$ to another system $x^{\beta'}$ in a Lorentz frame $\sum'$ which is moving with a uniform velocity ν in an arbitrary direction such that

$$x^{\beta'} = \wedge_\alpha^{\beta'} x^\alpha, x^\alpha = \wedge_{\beta'}^\alpha x^{\beta'} \tag{1.2}$$

where we have followed Einstein summation convention for repeated indices and the coefficient matrices are constant matrices which satisfy

$$\begin{array}{llll} \text{a)} & \wedge_\gamma^{\alpha'} \wedge_\delta^{\beta'} \eta_{\alpha'\beta'} = \eta_{\gamma\delta} & \text{b)} & \wedge_{\gamma'}^\alpha \wedge_{\delta'}^\beta \eta_{\alpha\beta} = \eta_{\gamma'\delta'} \\ \text{c)} & \wedge_\beta^{\alpha'} \wedge_{\gamma'}^\beta = \delta_{\gamma'}^{\alpha'} & \text{d)} & \wedge_{\beta'}^\alpha \wedge_\gamma^{\beta'} = \delta_\gamma^\alpha \end{array} \tag{1.3}$$

Note that $\eta_{\alpha'\beta'}$ are defined as in (1.1) by setting

$$\eta_{\alpha'\beta'} = \eta(e_{\alpha'}, e_{\beta'}) \ \ with \ \ e_{\alpha'} = \frac{\partial}{\partial x^{\alpha'}}.$$

Since the coefficient matrices in (1.2) are constants we have

$$dx^{\beta'} = \wedge_\alpha^{\beta'} dx^\alpha, \quad dx^\alpha = \wedge_\alpha^{\beta'} dx^{\beta'} \tag{1.4}$$

Hence

$$\eta_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'} = \eta_{\alpha'\beta'} \wedge_\gamma^{\alpha'} \wedge_\delta^{beta'} dx^\gamma dx^\delta = \eta_{\gamma\delta} dx^\gamma dx^\delta \tag{1.5}$$

We set

$$d\tau^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{1.6}$$

Then (1.5) shows that $d\tau$ is invariant under a Lorentz transformation. τ is called the proper time. The coordinate expression of a world line $X(\tau)$ of a particle is given by

$$X(\tau) = x^\alpha e_\alpha = x^{\beta'} e_{\beta'} \tag{1.7}$$

Since the basis vectors do not depend on the parameter τ, we have $dx^\alpha e_\alpha = dx^{\beta'} e_{\beta'}$. Using (1.4) we obtain

$$dx^{\beta'} e_{\beta'} = \wedge_\gamma^{\beta'} dx^\gamma e_{\beta'} = dx^\gamma e_\gamma.$$

Since the basis 1-forms dx^γ are linearly independent, we get

$$e_\gamma = \wedge_\gamma^{\beta'} e_{\beta'} \qquad e_{\gamma'} = \wedge_{\gamma'}^\beta e_\beta \tag{1.8}$$

The inner product in Σ is given by

$$\langle e_\alpha e_\beta \rangle = e_\alpha . e_\beta = \eta_{\alpha\beta},$$

and that in $\sum'$ is given by

$$\langle e_{\alpha'} e_{\beta'} \rangle = e_{\alpha'} . e_{\beta'} = \eta_{\alpha'\beta'},$$

$\eta_{\alpha\beta}$ and $\eta_{\alpha'\beta'}$ are related to each other as in (1.3a) and (1.3b).

The following lemma is well known. We give it and also the outlines of its proof in order to fix the notations and give physical meaning to the parameters occurring there in.

Lemma 1.1[5],[6]. A solution of the equation (1.2) is given by

$$\begin{aligned}
&\text{a) } \wedge_0^{0'} = \frac{1}{\sqrt{1-\nu^2}} = \Gamma, \Gamma \text{ is used for convenience}\\
&\text{b) } \wedge_0^{i'} = -\Gamma \nu n^i,\ \nu n^i = \nu^i,\ i = 1, 2, 3, \dagger\\
&\text{c) } \wedge_k^{j'} = (\Gamma - 1) n^j n_k + \delta_k^j \qquad (1.10)\\
&\text{d) } \wedge_j^{0'} = \wedge_0^{j'}\\
&\text{e) } \wedge_{\gamma'}^{\mu} = \wedge_\mu^{\gamma'} \text{ except that } \nu \text{ is changed into } -\nu
\end{aligned}$$

Proof. We show that the relations a) to e) are interdependent.

Setting $\gamma = \delta = 0$ in (1.3 a) and using (1.10 b) we get

$$\begin{aligned}
-1 &= (\wedge_0^{0'})^2 + \sum_{i=1}^{3} \Gamma^2 (-\nu n_i)^2\\
&= \Gamma^2(-1 + \nu^2)
\end{aligned}$$

which leads to (a).

For other $\wedge_k^{j'}$, there is no unique solution and (1.10 c) is found to be a convenient solution of (1.2). We use this to establish (d).

Set $\gamma = \delta = 1$ in (1.3 a) and use (1.10 c) to get

$$\begin{aligned}
1 &= -(\wedge_1^{0'})^2 + \sum_{i=1}^{3} (\wedge_1^{i'})^2\\
&= -(\wedge_1^{0'})^2 + [(\Gamma - 1) n_1^2 + 1]^2 + [(\Gamma - 1) n_1 n_2]^2 + [(\Gamma - 1) n_1 n_3]^2\\
&= -(\wedge_1^{0'})^2 + (\Gamma - 1)^2 n_1^2 + 2 + (\Gamma - 1) n_1^2 + 1\\
&= -(\wedge_1^{0'})^2 + n_1^2 [\Gamma^2 - 2\Gamma + 1 + 2\Gamma - 2] + 1\\
&= -(\wedge_1^{0'})^2 + n_1^2 [\frac{1}{1 - \nu^2} - 1] + 1\\
&= -(\wedge_1^{0'})^2 + n_1^2 \frac{\nu^2}{1 - \nu^2} + 1
\end{aligned}$$

† Greek letters $\alpha, \beta, \gamma \cdots$ takes values 0, 1, 2, 3 whereas Latin letters i, j, k take the value 1, 2, 3.

or,

$$\begin{aligned}-\Lambda_1^{0'} &= -\frac{n_1\nu}{\sqrt{1-\nu^2}} + 1 \\ &= -\Gamma n_1 \nu = \Lambda_0^{1'} \quad \text{from (b)}\end{aligned}$$

This leads to the conclusion $\Lambda_0^{j'} = \Lambda_j^{0'}$

Lemma 1.2. According to (e) in (1.10) we have

$$\begin{aligned}&\text{a) } \Lambda_{0'}^{i} = \Gamma\nu n_i = \Lambda_{i'}^{0} \\ &\text{b) } \Lambda_{j'}^{i} = \Lambda_j^{i'} \quad i,j = 1,2,3 \qquad (1.11) \\ &\text{c) } \Lambda_{0'}^{0} = \Gamma\end{aligned}$$

since the expressions for b) and c) contain Γ which is the same for both ν and $-\nu$. Further the relations in (1.3) are consistent with the equations in (1.10) and (1.11)

Proof 1. For instance setting $\gamma = \delta = 0$ in (1.3 a), we have

$$\Lambda_0^{\alpha'}\Lambda_0^{\beta'}\eta_{\alpha'\beta'} = -(\Lambda_{0'}^{0})^2 + \sum_{i=1}^{3}(\Lambda_0^{i'})^2 - \Gamma^2 + \Gamma^2\nu^2 = -1 = \eta_{00}$$

2. In (1.3 b) set $\gamma' = \delta' = 0$, then

$$-(\Lambda_{0'}^{0})^2 + \sum_{i=0}^{3}(\Lambda_{0'}^{i})^2 = -\Gamma^2 + \Gamma^2\nu^2 = -1 = \eta_{0'0'}$$

3. In (1.3 c) set $\alpha' = \gamma' = 0$, then

$$\begin{aligned}\Lambda_0^{0'}\Lambda_{0'}^{0} + \Lambda_1^{0'}\Lambda_{0'}^{1} + \Lambda_{0'}^{2}\Lambda_{0'}^{2} + \Lambda_3^{0'}\Lambda_{0'}^{3} &= \Gamma^2 - \Gamma^2\nu^2(n_1^2 + n_2^2 + n_3^2) \\ &= \Gamma^2 - \Gamma^2\nu^2 = 1 = \delta_{0'}^{0'}\end{aligned}$$

4. In (1.3 d) set $\alpha = \gamma = 1$ then

$$\Lambda_{0'}^{1}\Lambda_1^{0'} + \Lambda_{1'}^{1}\Lambda_1^{1'} + \Lambda_{2'}^{1}\Lambda^{2'}1 + \Lambda_{3'}^{1}\Lambda^{3'}1$$

$$\begin{aligned}&= -\Gamma^2\nu^2 n_1^2 + [(\Gamma-1)n_1^2 + 1]^2 + (\Gamma-1)^2 n_1^2 n_2^2 + (\Gamma-1)^2 n_1^2 n_3^2 \\ &= -\Gamma^2\nu^2 n_1^2 + (\Gamma-1)^2 n_1^2(n_1^2 + n_2^2 + n_3^2) + 2(\Gamma-1)n_1^2 + 1 \\ &= \Gamma^2 n_1^2(1-\nu^2) + n_1^2 - 2n_1^2 1 \\ &= 1\end{aligned}$$

2.Invariance of dV and $*dX$†

Lemma 2.1 The volume element in Minkowski space is invariant under a Lorentz transformation.

Proof. From (1.4) we have

$$dx^{0'} = \Lambda_0^{0'} dx^0 + \Lambda_j^{0'} dx^j, \qquad dx^{1'} = \Lambda_0^{1'} dx^0 + \Lambda_j^{1'} dx^j$$

$$dx^{2'} = \Lambda_0^{2'} dx^0 + \Lambda_l^{2'} dx^l, \qquad dx^{3'} = \Lambda_0^{3'} dx^0 + \Lambda_m^{3'} dx^m$$

From these we get

$$dx^{0'} \wedge dx^{1'} = B_j dx^0 \wedge dx^j + C_{jk}^{01} dx^j \wedge dx^k$$

$$dx^{2'} \wedge dx^{3'} = A_l dx^0 \wedge dx^l + D_{lm}^{23} dx^l \wedge dx^m$$

where

$$\text{a)} \quad B_j = \Lambda_0^{0'} \Lambda_j^{1'} - \Lambda_j^{0'} \Lambda_0^{1'} \qquad \text{b)} \quad C_{j\ k}^{0\ 1} = \Lambda_j^{0'} \Lambda_k^{1'} - \Lambda_k^{0'} \Lambda_j^{1'}$$

$$\text{c)} \quad A_m = \Lambda_0^{2'} \Lambda_m^{3'} - \Lambda_m^{2'} \Lambda_0^{3'} \qquad \text{d)} \quad D_{l\ m}^{2\ 3} = \Lambda_l^{2'} \Lambda_m^{3'} - \Lambda_m^{2'} \Lambda_l^{3'}$$

Hence

$$dV' = dx^{0'} \wedge dx^{1'} \wedge dx^{2'} \wedge dx^{3'} \ ‡$$

$$= B_j D_{l\ m}^{2\ 3} dx^0 \wedge dx^j \wedge dx^l \wedge dx^m + A_l C_{j\ k}^{0\ l} dx^0 \wedge dx^l \wedge dx^j \wedge dx^k$$

where the product term containing $C_{j\ k}^{0\ 1}$, $D_{l\ m}^{2\ 3}$ does not survive as it contains product of $dx^1, dx^2 \cdots$ which are repeated.

$$\text{I term} \quad = (B_1 D_{2\ 3}^{2\ 3} + B_2 D_{3\ 1}^{2\ 3} + B_3 D_{1\ 2}^{2\ 3}) dV$$
$$\text{II term} \quad = (A_1 C_{2\ 3}^{0\ 1} + A_2 C_{3\ 1}^{0\ 1} + A_3 C_{1\ 2}^{0\ 1}) dV$$

Now using (1.10), (1.11) and (2.1) we have

$$\begin{aligned} B_1 D_{2\ 3}^{3\ 2} &= (\Lambda_0^{0'} \Lambda_1^{1'} - \Lambda_1^{0'} \Lambda_0^{1'})(\Lambda_2^{2'} \Lambda_3^{3'} - \Lambda_3^{2'} \Lambda_2^{3'}) \\ &= [\Gamma(\Gamma-1)n_1^2 + \Gamma - \nu^2 \Gamma^2 n_1^2][((\Gamma-1)n_2^2+1)((\Gamma-1)n_3^2+1) - (\Gamma-1)^2 n_2^2 n_3^2] \\ &= [\Gamma^2(1-\nu^2)n_1^2 + \Gamma - \Gamma n_1^2][(\Gamma-1)(1-n_1^2)+1] \\ &= [n_1^2 + \Gamma(1-n_1^2)][\Gamma(1-n_1^2)+n_1^2] \\ &= [n_1^2(1-\Gamma)+\Gamma]^2 \end{aligned}$$

† For details of $*dX$, see appendix given at the end of this paper.

‡ In the appendix we have $dV = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0 = -dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. However this change in sign does not affect the invariance aspect

$$
\begin{aligned}
B_2 D_{3\ 1}^{2\ 3} &= (\wedge_0^{0'} \wedge_2^{1'} - \wedge_2^{0'} \wedge_0^{1'})(\wedge_3^{2'} \wedge_1^{3'} - \wedge_1^{2'} \wedge_3^{3'}) \\
&= [\Gamma(\Gamma-1)n_1 n_2 - \nu^2\Gamma^2 n_2 n_1]\{(\Gamma-1)^2 n_1 n_2 n_3^2 - (\Gamma-1)n_2 n_1[(\Gamma-1)n_3^2+1]\} \\
&= [\Gamma^2(1-\nu^2)n_1 n_2 - \Gamma n_1 n_2][-(\Gamma-1)n_1 n_2] \\
&= n_1^2 n_2^2 (1-\Gamma)^2
\end{aligned}
$$

$$
\begin{aligned}
B_3 D_{1\ 2}^{2\ 3} &= (\wedge_0^{0'} \wedge_3^{1'} - \wedge_3^{0'} \wedge_0^{1'})(\wedge_1^{2'} \wedge_2^{3'} - \wedge_2^{2'} \wedge_1^{3'}) \\
&= (\Gamma(\Gamma-1)n_1 n_3 - \Gamma^2\nu^2 n_1 n_3)\{(\Gamma-1)^2 n_1 n_3 n_2^2 - [(\Gamma-1)n_2^2+1](\Gamma-1)n_3 n_1\} \\
&= (n_1 n_3 - \Gamma n_1 n_3)[-(\Gamma-1)n_3 n_1] \\
&= n_1^2 n_3^2 (1-\Gamma)^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{I term} &= \left\{(n_1^2(1-\Gamma)+\Gamma)^2 + (1-\Gamma)^2 n_1^2 n_2^2 + (1-\Gamma)^2 n_1^2 n_3^2\right\} \\
&= \left\{[n_1^2(1-\Gamma)+\Gamma]^2 + (1-\Gamma)^2 n_1^2(1-n_1^2)\right\} dV \\
&= \left\{n_1^4(1-\Gamma)^2 + 2\Gamma(1-\Gamma)n_1^2 + \Gamma^2 + (1-\Gamma)^2 n_1^2 - (1-\Gamma)^2 n_1^4\right\} dV \\
&= \left\{2\Gamma(1-\Gamma)n_1^2 + \Gamma^2 + (1-\Gamma)^2 n_1^2\right\} dV \\
&= \left\{n_1^2(2\Gamma - 2\Gamma^2 + 1 - 2\Gamma + \Gamma^2) + \Gamma^2\right\} \\
&= \left\{n_1^2(1-\Gamma^2) + \Gamma^2\right\} dV
\end{aligned}
$$

Since $A_j = \Gamma\nu(n_3\delta_j^2 - n_2\delta_j^3)$ we have $A_1 = 0$, $A_2 = \Gamma\nu n_3$, $A_3 = -\Gamma\nu n_2$.

Further

$$
\begin{aligned}
C_{3\ 1}^{0\ 1} &= \wedge_3^{0'} \wedge_1^{1'} - \wedge_1^{0'} \wedge_3^{1'} \\
&= -\Gamma\nu\left\{n_3[(\Gamma-1)n_1^2+1] - (\Gamma-1)n_3 n_1^2\right\} \\
&= -\Gamma\nu n_3
\end{aligned}
$$

$$
\begin{aligned}
C_{1\ 2}^{0\ 1} &= \wedge_1^{0'} \wedge_2^{1'} - \wedge_2^{0'} \wedge_1^{1'} \\
&= \Gamma\nu\left\{(\Gamma-1)n_1^2 n_2 - n_2[(\Gamma-1)n_1^2+1]\right\} \\
&= \Gamma\nu n_2
\end{aligned}
$$

so

$$
\begin{aligned}
\text{II term} &= A_1 C_{2\ 3}^{0\ 1} + A_2 C_{3\ 1}^{0\ 1} + A_3 C_{1\ 2}^{0\ 1} \\
&= 0 + (-\Gamma\nu n_3)(\Gamma\nu n_3) + (\Gamma\nu n_2)(-\Gamma\nu n_2) \\
&= \Gamma^2\nu^2(-n_3^2 - n_2^2) \\
&= (n_1^2 - 1)\Gamma^2\nu^2 \\
\text{I term} + \text{II term} &= \left\{n_1^2 + \Gamma^2 - n_1^2\Gamma^2 - \Gamma^2\nu^2 + \Gamma^2\nu^2 n_1^2\right\} dV \\
&= \left\{n_1^2 + \Gamma^2(1-\nu^2) + n_1^2\Gamma^2(-1+\nu^2)\right\} dV \\
&= \left\{n_1^2 + 1 - n_1^2\right\} dV \\
&= dV
\end{aligned}
$$

Thus $dV' = dV$.

Lemma 2.2 Let $X' = x^{\alpha'} e_{\alpha'}$ and $X = x^{\alpha} e_{\alpha}$ be two coordinate systems in Minkowski space related to each other by a Lorentz transformation, then

$$*dX' = dX$$

In other words $*dX$ is invariant under a Lorentz transformation.

Proof. We have

$$*dX' = dx^{2'} \wedge dx^{3'} \wedge dx^{0'} e'_1 + dx^{3'} \wedge dx^{1'} \wedge dx^{0'} e'_2$$
$$+ dx^{1'} \wedge dx^{2'} \wedge dx^{0'} e'_3 + dx^{1'} \wedge dx^{2'} \wedge dx^{3'} e'_0$$

We show that the expression for $*dX'$ obtained in terms of unprimed quantities through a Lorentz transformation is precisely that of $*dX$. For this it is enough if we show that the coefficient of e_1 is $dx^2 \wedge dx^3 \wedge dx^0$ and that of e_0 is $dx^1 \wedge dx^2 \wedge dx^3$. As in Lemma 2.1, we have

$$dx^{2'} \wedge dx^{3'} = A_j dx^0 \wedge dx^j + A^{23}_{jk} dx^j \wedge dx^k$$

$$dx^{0'} = \Lambda^{0'}_0 dx^0 + \Lambda^{0'}_l dx^l$$

$$= \Gamma dx^0 + \Lambda^{0'}_l dx^l$$

Further using (1.8) we get

$$1.\ dx^{2'} \wedge dx^{3'} \wedge dx^{0'} e'_1 = \{(A_j \Lambda^{0'}_k - A_k \Lambda^{0'}_j + \Gamma A^{2}_{j}{}^{3}_{k}) dx^j \wedge dx^k \wedge dx^0 + A^{2}_{j}{}^{3}_{k} \Lambda^{0'}_l \, dx^j \wedge dx^k \wedge dx^l\} \Lambda^{\alpha}_{1'} e_\alpha \tag{2.2}$$

Similarly we have

$$2.\ dx^{3'} \wedge dx^{1'} \wedge dx^{0'} e'_2 = \{(B_j \Lambda^{0'}_k - B_k \Lambda^{0'}_j + \Gamma A^{3}_{j}{}^{1}_{k}) dx^j \wedge dx^k \wedge dx^0 + A^{3}_{j}{}^{1}_{k} \Lambda^{0'}_l \, dx^j \wedge dx^k \wedge dx^l\} \Lambda^{\alpha}_{2'} e_\alpha$$

$$3.\ dx^{1'} \wedge dx^{2'} \wedge dx^{0'} e'_3 = \{(C_j \Lambda^{0'}_k - C_k \Lambda^{0'}_j + \Gamma A^{1}_{j}{}^{2}_{k}) dx^j \wedge dx^k \wedge dx^0 + A^{1}_{j}{}^{2}_{k} \Lambda^{0'}_l \, dx^j \wedge dx^k \wedge dx^l\} \Lambda^{\alpha}_{3'} e_\alpha$$

$$4.\ dx^{1'} \wedge dx^{2'} \wedge dx^{3'} e'_0 = \{(C_j \Lambda^{3'}_k - C_k \Lambda^{3'}_j + \Gamma A^{1}_{j}{}^{2}_{k} \Lambda^{3'}_0) dx^j \wedge dx^k \wedge dx^0 + A^{1}_{j}{}^{2}_{k} \Lambda^{3'}_l \, dx^j \wedge dx^k \wedge dx^l\} \Lambda^{\alpha}_{0'} e_\alpha$$

where

$$
\begin{aligned}
&1.\ A_j^1{}_k^2 = \wedge_j^{1'}\wedge_k^{2'} - \wedge_k^{1'}\wedge_j^{2'} = (\Gamma-1)n_1n_j\delta_k^2 + (\Gamma-1)n_2n_k\delta_j^1\\
&\qquad -(\Gamma-1)n_1n_k\delta_j^2 - (\Gamma-1)n_2n_j\delta_k^1 + \delta_j^1\delta_k^2 - \delta_k^1\delta_j^2\\
&2.\ A_j^2{}_k^3 = \wedge_j^{2'}\wedge_k^{3'} - \wedge_k^{2'}\wedge_j^{3'} = (\Gamma-1)n_2n_j\delta_k^3 + (\Gamma-1)n_3n_k\delta_j^2\\
&\qquad -(\Gamma-1)n_2n_k\delta_j^3 - (\Gamma-1)n_3n_j\delta_k^2 + \delta_j^2\delta_k^3 - \delta_k^2\delta_j^3\\
&3.\ A_j^3{}_k^1 = \wedge_j^{3'}\wedge_k^{1'} - \wedge_k^{3'}\wedge_j^{1'} = (\Gamma-1)n_3n_j\delta_k^1 + (\Gamma-1)n_1n_k\delta_j^3\\
&\qquad -(\Gamma-1)n_3n_k\delta_j^1 - (\Gamma-1)n_1n_j\delta_k^3 + \delta_j^3\delta_k^1 - \delta_k^3\delta_j^1
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
&1.\quad A_j = \wedge_0^{2'}\wedge_j^{3'} - \wedge_j^{2'}\wedge_0^{3'} = \nu\Gamma(n_3\delta_j^2 - n_2\delta_j^3)\\
&2.\quad B_j = \wedge_0^{3'}\wedge_j^{1'} - \wedge_j^{3'}\wedge_0^{1'} = \nu\Gamma(n_1\delta_j^3 - n_3\delta_j^1)\\
&3.\quad C_j = \wedge_0^{1'}\wedge_j^{2'} - \wedge_j^{1'}\wedge_0^{2'} = \nu\Gamma(n_2\delta_j^1 - n_1\delta_j^2)
\end{aligned}
\tag{2.4}
$$

Hence the coefficient of e_1

$$
\begin{aligned}
&= \{\wedge_{1'}^1[(A_j\wedge_k^{0'} - A_k\wedge_j^{0'}) + \Gamma A_j^2{}_k^3] + \wedge_{2'}^1[(B_j\wedge_k^{0'} - B_k\wedge_j^{0'}) + A_j^3{}_k^1]\\
&\quad + \wedge_{3'}^1[(C_j\wedge_k^{0'} - C_k\wedge_j^{0'}) + \Gamma A_j^1{}_k^2] + \wedge_{0'}^1[(C_j\wedge_k^{0'} - C_k\wedge_j^{0'})\\
&\quad + A_j^1{}_k^2\wedge_0^{3'}]\}dx^j\wedge dx^k\wedge dx^0\\
&\quad + \{\wedge_l^{0'}(\wedge_{1'}^1 - A_j^2{}_k^3 + \wedge_{2'}^1 A_j^3{}_k^1 + \wedge_{3'}^1 A_j^1{}_k^2) + A_j^1{}_k^2\wedge_l^{3'}\cdot\wedge_{0'}^l\}dx^j\wedge dx^k\wedge dx^l
\end{aligned}
$$

$$
\begin{aligned}
\text{I term } &= \{\wedge_k^{0'}(\wedge_{1'}^1, A_j + \wedge_{2'}^1 B_j + \wedge_{3'}^1 C_j) - \wedge_j^{0'}(\wedge_{1'}^l A_k + \wedge_{2'}^1 B_k + \wedge_{3'}^1 C_k)\\
&\quad + \Gamma(\wedge_{l'}^1 A_j^2{}_k^3 + \wedge_{2'}^1 A_j^3{}_k^1 + \wedge_{3'}^1 A_j^1{}_k^2)\\
&\quad + \wedge_{0'}^1(C_j\wedge_k^{3'} - C_k\wedge_j^{3'} + \wedge_0^{3'}A_j^1{}_k^2)\}dx^j\wedge dx^k\wedge dx^0\\
&= \{\Gamma^2\nu^2[(n_2\delta_j^3 - n_3\delta_j^2)n_k - (n_2\delta_k^3 - n_3\delta_k^2)n_j]\\
&\quad + \Gamma(\Gamma-1)[n_2n_j\delta_k^3 + n_3n_k\delta_j^2 - n_2n_k\delta_j^3 - n_3n_j\delta_k^2]\\
&\quad + \Gamma(\delta_k^3\delta_j^2 - \delta_j^3\delta_k^2) + \Gamma(\Gamma-1)n_1^2(\delta_k^3\delta_j^2 - \delta_j^3\delta_k^2)\\
&\quad + \Gamma(\Gamma-1)n_1n_3(\delta_j^1\delta_k^2 - \delta_k^1\delta_j^2) + \Gamma(\Gamma-1)n_1n_2(\delta_j^3\delta_k^1 - \delta_k^3\delta_j^1)\\
&\quad + \Gamma^2\nu^2(n_1n_2\delta_j^1\delta_k^3 - n_1n_2\delta_k^1\delta_j^3 - n_1^2\delta_j^2\delta_k^3 + n_1^2\delta_k^2\delta_j^3)\}dx^j\wedge dx^k\wedge dx^0
\end{aligned}
$$

Setting $j = 2, k = 3$ we get the coefficient of $dx^2\wedge dx^3\wedge dx^0$

$$
\begin{aligned}
&= \Gamma^2\nu^2(-n_3^2 - n_2^2 - n_1^2) + \Gamma(\Gamma-1)(n_2^2 + n_3^2 + n_1^2) + \Gamma\\
&= -\Gamma^2\nu^2 + \Gamma^2 = 1
\end{aligned}
$$

Setting $j = 1, k = 1$ it is easy to see that the coefficient of $dx^1\wedge dx^2\wedge dx^0$ is zero. Similarly the coefficient of $dx^3\wedge dx^1\wedge dx^0$ is also seen to be zero.

Thus we have coefficient of $e_1 = dx^2 \wedge dx^3 \wedge dx^0 +$ II term
We show that II term vanishes.

II term $\{\wedge_l^{0'}(A_j^2{}_k^3\wedge_{1'}^1 + A_j^3{}_k^1\wedge_{2'}^1 + A_j^1{}_k^2\wedge_{3'}^1) + \wedge_{0'}^1 A_j^1{}_k^2\wedge_l^{3'}\}dx^j\wedge dx^k\wedge dx^l$

$$= \{\wedge_l^{0'}[(\Gamma-1)(n_2n_j\delta_k^3 + n_3n_k\delta_j^2 - n_2^k\delta_j^3 - n_3n_j\delta_k^2)$$
$$+ (\delta_j^2\delta_k^3 - \delta_k^2\delta_j^3) + (\Gamma-1)n_1^2(\delta_j^2\delta_k^3 - \delta_k^2\delta_j^3) + (\Gamma-1)n_1n_2(\delta_j^3\delta_k^1 - \delta_k^3\delta_j^1)$$
$$+ (\Gamma-1)n_1n_3(\delta_j^1\delta_k^2 - \delta_k^1\delta_j^2)\}dx^j\wedge dx^k\wedge dx^l$$
$$+ \wedge_{0'}^1(A_{1'}^1{}_2^2\wedge_3^{3'} + A_{2'}^1{}_3^2\wedge_1^{3'} + A_{3'}^1{}_1^2\wedge_2^{3'})dx^1\wedge dx^2\wedge dx^3$$
$$= \{\wedge_l^{0'}(\Gamma-1)(-n_3n_1 + n_3n_1)dx^1\wedge dx^2\wedge dx^l$$
$$+ \wedge_l^{0'}[(\Gamma-1)(n_2^2+n_3^2) + 1 + (\Gamma-1)n_1^2]dx^2\wedge dx^3\wedge dx^l$$
$$+ \wedge_l^{0'}(\Gamma-1)(-n_2n_1 + n_2n_1)\}dx^3\wedge dx^1\wedge dx^l$$
$$+ \wedge_{0'}1\{[(\Gamma-1)(n_1^2+n_2^2)+1][(\Gamma-1)n_3^2+1] + [-(\Gamma-1)n_1n_3](\Gamma-1)n_1n_3$$
$$+ [-(\Gamma-1)n_2n_3](\Gamma-1)n_2n_3\}dx^1\wedge dx^2\wedge dx^3$$
$$= \wedge_l^{0'}\Gamma dx^2\wedge dx^3\wedge dx^l + \wedge_{0'}^1\{[(\Gamma-1)(n_1^2+n_2^2)+1][(\Gamma-1)n_3^2+1]$$
$$- (\Gamma-1)^2n_1^2n_3^2 - (\Gamma-1)n_2^2n_3^2\}dx^1\wedge dx^2\wedge dx^3$$
$$= \wedge_1^{0'}\Gamma dx^2\wedge dx^3\wedge dx^1 + \wedge_{0'}^1[(\Gamma-1)(n_1^2+n_2^2) + (\Gamma-1)n_3^2 + 1]dx^1\wedge dx^2\wedge dx^3$$
$$= -\Gamma^2\nu n_1 dx^1\wedge dx^2\wedge dx^3 + \Gamma\nu n_1(\Gamma-1+1)dx^1\wedge dx^2\wedge dx^3$$
$$= 0$$

The coefficient of e_0 is

$$= \{\wedge_k^{0'}\wedge_{1'}^0 A_j + \wedge_{2'}^0 B_j + \wedge_{3'}^0 C_j) - \wedge_j^{0'}(\wedge_{1'}^0 A_k + \wedge_{2'}^0 B_k + \wedge_{3'}^0 C_k)$$
$$+ \Gamma(\wedge_0^{1'}A_j^2{}_k^3 + \wedge_0^{2'}A_j^3{}_k^1 + \wedge_0^{3'}A_j^1{}_k^2) + \wedge_{0'}^0(C_j\wedge_k^{3'} - C_k\wedge_j^{3'}) + \wedge_0^{3'}A_j^1{}_k^2\}dx^j\wedge dx^k\wedge dx^0$$
$$+ \{(\wedge_l^{0'}(A_j^2{}_k^3\wedge_0^{1'} + A_j^3{}_k^1\wedge_0^{2'} + A_j^1{}_k^2\wedge_{3'}^0) + \wedge_{0'}^0(A_j^1{}_k^2\wedge_{3'}^l\}dx^j\wedge dx^k\wedge dx^l$$
$$= \text{I term} + \text{II term}$$

Here in I term

$$\wedge_{1'}^0 A_j + \wedge_{2'}^0 B_j + \wedge_{3'}^0 C_j$$
$$= \Gamma^2\nu^2[n_1n_3\delta_j^2 - n_1n_2\delta_j^3 + n_2n_1\delta_j^3 - n_2n_3\delta_j^1 + n_3n_2\delta_j^1 - n_3n_1\delta_j^2] = 0$$

Similarly $\wedge_{1'}^0 A_k + \wedge_{2'}^0 B_k + \wedge_{3'}^0 C_k = 0$

$$\Gamma(\wedge_0^{1'}A_j^2{}_k^3 + \wedge_{2'}^0 A_j^3{}_k^1 + \wedge_{3'}^0 A_j^1{}_k^2)$$
$$= \Gamma^2\nu\{n_1(\delta_k^3\delta_j^2 - \delta_j^3\delta_k^2) + n_2(\delta_j^3\delta_k^1 - \delta_k^3\delta_j^1) + n_3(\delta_j^1\delta_k^2 - \delta_k^1\delta_j^2)\}$$

$$\Gamma\{(C_j \wedge_k^{3'} - C_k \wedge_j^{3'} + \wedge_j^{3'} A_j{}^1{}_k^2\}$$

$$= \Gamma\{\Gamma\nu(n_2\delta_j^1 - n_1\delta_j^2)[(\Gamma-1)n_3n_k + \delta_k^3]$$
$$- \Gamma\nu(n_2\delta_k^1 - n_1\delta_k^2)[(\Gamma-1)n_3n_j + \delta_j^3]$$
$$- \Gamma\nu n_3[(\Gamma-1)n_1n_j\delta_k^2 + (\Gamma-1)n_2n_k\delta_j^1 - (\Gamma-1)n_1n_k\delta_j^2]$$
$$- (\Gamma-1)n_2n_j\delta_k^1 + \delta_j^1\delta_j^2 - \delta_k^1\delta_j^1)\}$$
$$= \Gamma^2\nu\{(n_1(\delta_k^2\delta_j^3 - \delta_j^2\delta_k^3) + n_2(\delta_k^3\delta_j^1 - \delta_j^3\delta_k^1) + n_3(\delta_k^1\delta_j^2 - \delta_j^1\delta_k^2)\}$$

Adding all the terms above we find that, I term = 0

$$\text{II term} = \{\wedge_l^{0'}(A_j^2{}_k^3 \wedge_{1'}^0 + A_j^3{}_k^1 \wedge_{2'}^0 + A_j^1{}_k^2\wedge_{3'}^0) + \wedge_{0'}^0 A_j^1{}_k^2 \wedge_l^{3'}\}dx^j \wedge dx^k \wedge dx^l$$

$$= \{\wedge_3^{0'}(A_1^2{}_2^3 \wedge_{1'}^0 + A_1^3{}_2^1 \wedge_{2'}^0 + A_1^1{}_2^2\wedge_{3'}^0) + \wedge_1^{0'}(A_3^2{}_1^3 \wedge_{1'}^0 + A_3^3{}_1^1 \wedge_{2'}^0 + A_3^1{}_1^2\wedge_{3'}^0)$$
$$+ \wedge_2^{0'}(A_2^2{}_3^3 \wedge_{1'}^0 + A_2^3{}_3^1 \wedge_{2'}^0 + A_2^1{}_3^2\wedge_{3'}^0) + \wedge_{0'}^0(A_1^1{}_2^2 \wedge_3^{3'} + A_2^1{}_3^2 \wedge_1^{3'} + A_3^1{}_1^2\wedge_2^{3'})\}$$

$$\wedge_3^{0'}(A_1^2{}_2^3 \wedge_{1'}^0 + A_1^3{}_2^1 \wedge_{2'}^0 + A_1^1{}_2^2\wedge_{3'}^0)$$
$$= -\Gamma\nu n_3\{[-\Gamma(\Gamma-1)n_1^2n_3 - \Gamma(\Gamma-1)n_2^2n_3] + \Gamma\nu n_3[(\Gamma-1)n_1^2 + (\Gamma-1)n_2^2 + 1]\}$$
$$= -\Gamma^2\nu^2 n_3^2$$

Similarly the term containing $\wedge_2^{0'}$ is $-\Gamma^2\nu^2n_2^2$ and that containing $\wedge_1^{0'}$ is $-\Gamma^2\nu^2n_1^2$. Also the term containing $\wedge_{0'}^0$ is Γ^2 (for computations see that of the last term namely the coefficient of $\wedge_{0'}^1$ in II term above).
So the coefficient of e_0 is

$$= [-\Gamma^2\nu^2(n_1^2 + n_2^2 + n_3^2) + \Gamma^2]dx^1 \wedge dx^2 \wedge dx^3$$
$$= \Gamma^2(1-\nu^2)dx^1 \wedge dx^2 \wedge dx^3$$
$$= dx^1 \wedge dx^2 \wedge dx^3.$$

3. Differential operators in Minkowski space

In the appendix given at the end of this paper we have

$$dx^\alpha \wedge *dx^\beta = \eta^{\alpha\beta}dV \tag{3.1}$$

where

$$dV = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^0$$

Also,

$$*dX = *dx^\alpha e_\alpha \tag{3.2}$$

We have shown that both dV and $*dX$ are invariant with respect to Lorentz transformation. We use these in the study of differential operators.

Lemma 3.1 Let f be a differential real valued function and F, a differentiable vectorfield on (M,η), then

$$\begin{aligned}
&(a)\quad df\wedge *dX &&= (grad\, f)dV = \eta^{\alpha\beta}\tfrac{\partial f}{\partial x^\alpha}e_\beta dV\\
&(b)\quad dF\cdot\wedge * dX &&= (div\, F)dV = \frac{\partial F^\mu}{\partial x^\mu}dV \qquad (3.3)\\
&(c)\quad d(grad\, f)\cdot\wedge * dX &&= \nabla f dV
\end{aligned}$$

$\Box f$ is known as Laplacian of f and $\cdot$ indicates dot product between vectors.

Proof (a)

$$\begin{aligned}
df\wedge *dX &= \frac{\partial f}{\partial x^\alpha}dx^\alpha\wedge *dx^\beta e_\beta\\
&= \eta^{\alpha\beta}\frac{\partial f}{\partial x^\alpha}e_\beta dV\\
&= (grad\, f)dV
\end{aligned}$$

If we use primed coordinate system, we obtain

$$df\wedge *dX' = \eta^{\alpha'\beta'}\frac{\partial f}{\partial x^{\alpha'}}e_{\beta'}dV = (grad\, f)'dV'$$

Using the invariance of dV and $*dX$, we get

$$df\wedge *dX' = (grad\, f)'dV' = df\wedge *dX = (grad\, f)dV$$

Since $dV = dV'$ we have $grad\, f = (grad\, f)'$

(b) Let $F = F^\alpha e_\alpha$ be a differentiable vector field in (M,η). The $dF = dF^\alpha e_\alpha$ and

$$\begin{aligned}
dF\wedge\cdot * dX &= \frac{\partial F^\mu}{\partial x^\beta}dx^\beta\wedge *dx^\alpha\langle e_\alpha, e_\mu\rangle dV\\
&= \frac{\partial F^\alpha}{\partial x^\beta}\eta^{\beta\alpha}\eta_{\alpha\mu}dV\\
&= \frac{\partial F^\mu}{\partial x^\mu}dV\\
&= (div\, F)dV
\end{aligned}$$

Following the procedure used in (a) above, it is easy to show that $divF$ does not depend on any particular frame. Using $\cdot$ to indicate dot product between vectors, we have

(c)

$$\begin{aligned}
d(grad\, f)\cdot\wedge * dX &= \eta^{\alpha\beta}d(\frac{\partial f}{\partial x^\alpha}e_\beta\cdot\wedge * dx^\mu e_\mu\\
&= \eta^{\alpha\beta}\eta_{\mu\beta}(\frac{\partial^2 f}{\partial x^\alpha\partial x\gamma}dx^\gamma\wedge *dx^\mu\\
&= \eta^{\mu\gamma}\frac{\partial^2 f}{\partial x^\alpha\partial x^\gamma}\delta^\alpha_\mu dV\\
&= \Box f dV
\end{aligned}$$

Since (c) is a special case of (b), it is clear that $\Box f$ also does not depend on any particular frame.

Definition 3.1 Let F be a differentiable vectorfield on (M,η).$curl\,F$ is given by

$$(curl\,F)dV = -dF \wedge\wedge * dX \tag{3.4}$$

where the two wedge product symbols are used to indicate both the wedge product between differential forms and the vector product between vectors.

Lemma 3.2 We have

$$curl\,F = \frac{\partial F^\alpha}{\partial x^\beta}\eta^{\beta\gamma}e_\gamma \wedge e_\alpha \tag{3.5}$$

Proof.

$$\begin{aligned} -dF \wedge\wedge * dX &= -\frac{\partial F^\alpha}{\partial x^\beta}dx^\beta e_\alpha \wedge\wedge * dx^\gamma e_\gamma \\ &= \frac{\partial F^\alpha}{\partial x^\beta}\eta^{\gamma\beta}e_\gamma \wedge e_\alpha dV \\ &= (curl\,F)dV \end{aligned}$$

from which the result follows.

Corollary 3.3 We have

$$curl(grad\,f) = 0$$

for every differentiable real valued function f defined on (M,η).

Proof.

$$\begin{aligned} curl(grad\,f) &= \frac{\partial}{\partial x^\beta}(\eta^{\alpha\delta}\frac{\partial f}{\partial x^\delta})\eta_{\beta\gamma}e_\gamma \wedge e_\alpha \\ &= \eta^{\alpha\delta}\eta^{\beta\gamma}\frac{\partial^2 f}{\partial x^\beta \partial x^\delta}e_\gamma \wedge e_\alpha \end{aligned}$$

Since $\frac{\partial^2 f}{\partial x^\beta \partial x^\delta} = \frac{\partial^2 f}{\partial x^\delta \partial x^\beta}$, we have

$$curl(grad\,f) = \frac{1}{2}(\eta^{\alpha\delta}\eta^{\beta\gamma} + \eta^{\alpha\beta}\eta^{\delta\gamma})\frac{\partial^2 f}{\partial x^\beta \partial x^\delta}e_\gamma \wedge e_\alpha = 0$$

Corollary 3.4 Let $\bar{F} = \sum_{j=1}^{3} F^j e_j$ be the space component of a vectorfield F on a (M,η). Then

$$curl\,F = (grad\,F^0 + \frac{\partial \bar{F}}{dx^0}) \wedge e_0 + (curl\,\bar{F})_3 \tag{3.6}$$

where $(curl\,\bar{F})_3$ denotes the curl of Euclidean vector field $\bar{F}$ in R^3.

Proof. Let $F = F^0 e_0 + F^j \epsilon_j = F^0 e_0 + \bar{F}$

$$\begin{aligned} curl\,F &= \frac{\partial F^0}{\partial x^\beta}\eta^{\beta\gamma}e_\gamma \wedge e_0 + \frac{\partial F^j}{\partial x^\beta}\eta^{\beta\gamma}e_\gamma \wedge e_j \quad j=1,2,3. \\ &= (grad\,F^0)\wedge e_0 + \frac{\partial F^j}{\partial x^0}e_j \wedge e_0 + \frac{\partial F^j}{\partial x^k}\eta^{k\gamma}e_\gamma \wedge e_j \\ &= (grad\,F^0 + \frac{\partial \bar{F}}{\partial x^0})\wedge e_0 + \frac{\partial F^j}{\partial x^k}\eta^{kl}e_l \wedge e_j \ \text{ since } \eta^{k0}=0,\ \ k=1,2,3 \\ &= (grad\,F^0 + \frac{\partial \bar{F}}{\partial x^0})\wedge e_0 + (curl\,\bar{F})_3 \end{aligned}$$

Note:

$$\begin{aligned} (curl\,\bar{F})_3 &= \eta^{kl}\frac{\partial F^j}{\partial x^k}e_l \wedge e_j \\ &= \eta^{11}\frac{\partial F^j}{\partial x^1}e_1 \wedge e_j + \eta^{22}\frac{\partial F^j}{\partial x^2}e_2 \wedge e_j + \eta^{33}\frac{\partial F^j}{\partial x^3}e_3 \wedge e_j \\ &= \frac{\partial F^2}{\partial x^1}e_1 \wedge e_2 + \frac{\partial F^3}{\partial x^1}e_1 \wedge e_3 + \frac{\partial F^1}{\partial x^2}e_2 \wedge e_1 + \frac{\partial F^3}{\partial x^2}e_2 \wedge e_3 \\ &\quad + \frac{\partial F^1}{\partial x^3}e_3 \wedge e_1 + \frac{\partial F^2}{\partial x^3}e_3 \wedge e_2 \\ &= \left(\frac{\partial F^2}{\partial x^1} - \frac{\partial F^1}{\partial x^2}\right)e_1 \wedge e_2 + \left(\frac{\partial F^3}{\partial x^2} - \frac{\partial F^2}{\partial x^3}\right)e_2 \wedge e_3 + \left(\frac{\partial F^1}{\partial x^3} - \frac{\partial F^3}{\partial x^1}\right)e_3 \wedge e_1 \end{aligned}$$

This is the usual expression for $curl\bar{F}$ in R^3, and

Expressions for $grad\,f$, $div\,F$ and $curl\,F$ in terms of spherical polar coordinates

If X is expressed in spherical polar coordinates, we have

$$X = x^0 e_0 + r\sin\theta\cos\phi e_1 + r\sin\theta\sin\phi e_2 + r\cos\theta e_3.$$

so that

$$dX = dx^0 e_0 + dre_r + rd\theta e_\theta + r\sin\theta d\phi e_\phi$$

where $e_r = \sin\theta\cos\phi e_1 + \sin\theta\cos\phi e_2 + \cos\theta e_3$

$$e_\theta = \cos\theta\cos\phi e_1 + \cos\theta\sin\phi e_2 - \sin\theta e_3, \quad e_\phi = -\sin\phi e_1 + \cos\phi e_2 \tag{3.7}$$

We write

$$\widehat{e}_0 = e_0,\ \widehat{e}_1 = e_r,\ \widehat{e}_2 = e_\theta,\ \widehat{e}_3 = e_\phi \tag{3.8}$$

Then it is easy to see that $(\widehat{e}_\alpha)$ is an orthonormal basis of the tangent space and the dual 1-forms are given by

$$\begin{aligned} &\omega^0 = dx^0,\ \omega^1 = dr,\ \omega^r = rd\theta,\ \omega^3 = r\sin\theta d\phi \\ &*\omega^0 = r^2\sin\theta dr \wedge d\theta \wedge d\phi, \quad *\omega^1 = r^2\sin\theta d\theta \wedge d\phi \wedge dx^0 \\ &*\omega^2 = r\sin\theta d\phi \wedge dr \wedge dx^0, \ *\omega^3 = rdr \wedge d\theta \wedge dx^0, \ *\omega^\alpha \wedge *\omega^\beta = \eta^{\alpha\beta}dV \end{aligned} \tag{3.9}$$

$$dV = \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^0 = r^2 \sin\theta dr \wedge dtheta \wedge d\phi \wedge dx^0 \tag{3.10}$$

Lemma 3.3 We have

$$\begin{aligned}
&\text{(a) } grad\, f &&= -\frac{\partial f}{\partial x^0}e_0 + \frac{\partial f}{\partial r}e_r + \frac{1}{r}\frac{\partial f}{\partial \theta}e_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}e_\phi \\
&\text{(b) } div\, F &&= \frac{\partial F^0}{\partial x^0} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F^r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F^\theta) \\
& &&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial F^\phi}{\partial \phi} \\
&\text{(c) } \Box f &&= \text{Laplacian of } f = -\frac{\partial^2 f}{\partial x^{0^2}} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial \theta}) \\
& &&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} \\
&\text{(d) } curl\, F &&= \left[(grad\, F^0) + \frac{\partial \bar{F}}{\partial x^0}\right] \wedge e_0 \\
& &&\quad + \left[\frac{1}{r}\frac{\partial}{\partial r}(rF^\theta) - \frac{1}{r}\frac{\partial F^r}{\partial \theta}\right] e_r \wedge e_\theta \\
& &&\quad + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(F^\phi \sin\theta) - \frac{\partial F^\theta}{\partial \phi}\right) e_\theta \wedge e_\phi \\
& &&\quad + \left(\frac{1}{r\sin\theta}(\frac{\partial F^r}{\partial \phi} - \frac{1}{r})\frac{\partial}{\partial r}(rF^\phi)\right) e_\phi \wedge e_r
\end{aligned} \tag{3.11}$$

Proof. **(a)** If f is differentiable real valued function on (M, η), then

$$df = \frac{\partial f}{\partial x^0}dx^0 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi})d\phi = f_\alpha \omega^\alpha$$

where

$$f_0 = \frac{\partial f}{\partial x^0}, \quad f_1 = \frac{\partial f}{\partial r}, \quad f_2 = \frac{1}{r}\frac{\partial f}{\partial \theta}, \quad f_3 = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}) \tag{3.12}$$

and ω^α are as given in (3.9).

Hence we have

$$(grad\, f)dV = df \wedge *dX = f_\alpha \omega^\alpha \wedge *\omega^\beta \widehat{e}_\beta = \eta^{\beta\alpha} f_\alpha \widehat{e}_\beta dV \quad \text{by (3.10)}$$

Or,

$$grad\, f = \eta^{\beta\alpha} f_\alpha \widehat{e}_\beta = -\frac{\partial f}{\partial x^0}\widehat{e}_0 + \frac{\partial f}{\partial r}\widehat{e}_1 + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{e}_2 + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\widehat{e}_3 \tag{3.13}$$

(b) Let $F = F^0 e_0 + F^r e_r + F^\theta e_\theta + F^\phi e_\phi$. Then

$$dF = dF^0 e_0 + dF^r e_r + dF^\theta e_\theta + dF^\phi e_\phi + F^r de_r + F^\theta de_\theta + F^\phi de_\phi.$$

From (3.7) it is easy to show that

$$de_r = d\theta e_\theta + \sin\theta d\phi e_\phi$$

$$de_\theta = -d\theta e_r + \cos\theta d\phi e_\phi \tag{3.14}$$

Hence

$$\begin{aligned} dF &= dF^0 e_0 + (dF^r - F^\theta d\theta - F^\phi \sin\theta\theta d\phi)e_r \\ &\quad +(dF^\theta + F^\phi d\theta - F^\phi \cos\theta d\phi)e_\theta \\ &\quad +(F^r \sin\theta d\phi + F^\theta \cos\theta d\phi + dF^\phi)e_\phi = dF^\beta \widehat{e}_\beta \end{aligned} \tag{3.15}$$

Writing $dF = dF^0\widehat{e}_0 + dF^1\widehat{e}_1 + dF^2\widehat{e}_2 + dF^3\widehat{e}_3$, where $\widehat{e}_i$ are as given in (3.8). We have

$$\begin{aligned} (div\, F)dV &= dF^\alpha e_\alpha \cdot \wedge * \omega^\beta e_\beta \\ &= dF^\alpha \wedge *\omega^\beta \langle e_\alpha, e_\beta\rangle \\ &= \eta_{\alpha\beta} dF^\alpha \wedge *\omega^\beta \end{aligned}$$

Here

$$\eta_{00} dF^0 \wedge *\omega^0 = -1 \cdot \frac{\partial F^0}{\partial x^0} dx^0 \wedge r^2 \sin\theta dr \wedge d\theta \wedge d\phi = \frac{\partial F^0}{\partial x^0} dV$$

$$\begin{aligned} \eta_{11} dF^1 \wedge *\omega^1 &= (dF^r - F^\theta d\theta - F^\phi \sin\theta) \wedge r^2 \sin\theta \wedge d\phi \wedge dx^0 \\ &= r^2 \sin\theta \frac{\partial F^r}{\partial r} dr \wedge d\theta \wedge d\phi \wedge dx^0 \\ &= \frac{\partial F^r}{\partial r} dV \end{aligned}$$

$$\begin{aligned} \eta_{22} dF^2 \wedge *\omega^2 &= (dF^\theta + F^r d\theta - F^\phi \cos\theta d\phi) \wedge r \sin\theta d\phi \wedge dr \wedge dx^0 \\ &= \frac{1}{r}\Big(\frac{\partial F^\theta}{\partial\theta} + F^r\Big) r^2 \sin\theta d\theta \wedge d\phi \wedge dr \wedge dx^0 \\ &= \frac{1}{r}\Big(\frac{\partial F^\theta}{\partial\theta} + F^r\Big) r^2 \sin\theta dr \wedge d\theta \wedge d\phi \wedge dx^0 \\ &= \frac{1}{r}\Big(\frac{\partial F^\theta}{\partial\theta} + F^r\Big) dV \end{aligned}$$

$$\begin{aligned} \eta_{33} dF^3 \wedge *\omega^3 &= (dF^r \sin\theta d\phi + F^\theta \cos\theta d\phi + dF^\phi) \wedge r dr \wedge d\theta \wedge dx^0 \\ &= \Big(\frac{F^r}{r} + \frac{F^\theta \cos\theta}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial F^\phi}{\partial\phi}\Big) r^2 \sin\theta d\phi \wedge dr \wedge d\theta \wedge dx^0 \\ &= \Big(\frac{F^r}{r} + \frac{F^\theta \cos\theta}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial F^\phi}{\partial\phi}\Big) r^2 \sin\theta dr \wedge d\theta \wedge d\phi \wedge dx^0 \\ &= \Big(\frac{F^r}{r} + \frac{F^\theta \cos\theta}{r\sin\theta} + \frac{1}{r\sin\theta}\frac{\partial F^\phi}{\partial\phi}\Big) dV \end{aligned}$$

$$\begin{aligned} div\, F &= \Big(\frac{\partial F^0}{\partial x^0} + \frac{2F^r}{r} + \frac{\partial F^r}{\partial r} + \frac{1}{r}\frac{\partial F^\theta}{\partial\theta} + \frac{\cos\theta}{r\sin\theta} F^\theta + \frac{1}{r\sin\theta}\frac{\partial F^\phi}{\partial\phi}\Big) \\ &= \Big(\frac{\partial F^0}{\partial x^0} + \frac{1}{r^2}\frac{\partial(r^2 F^r)}{dr} + \frac{1}{r\sin\theta}\frac{\partial(F^\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F^\phi}{\partial\phi}\Big) \end{aligned}$$

(c) By setting $F^0 = -\dfrac{\partial f}{\partial x^0}$, $F^r = \dfrac{\partial f}{\partial r}$, $F^\theta = \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}$, $F^\phi = \frac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi}$ in the formula for $div\, F$ given above we get

$$\begin{aligned}\Box f &= div(grad\, f) = \text{Laplacian of } f \\ &= -\frac{\partial^2 f}{\partial x^{0 2}} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}\end{aligned}$$

This result may be compared with the formula for Laplacian of f given in [1] on page 76.

(d) As in (3.15) writing $dF = dF^\beta \widehat{e}_\beta$ we have

$$\begin{aligned}-(curl\, F)dV &= dF \wedge \wedge * dX \\ &= dF^0\widehat{e}_0 \wedge\wedge * dX + dF^i\widehat{e}_i \wedge\wedge * dX \\ &= dF^0 \wedge\wedge * dX \wedge e_0 + d\bar{F} \wedge\wedge * \omega^0 e_0 + dF^i\widehat{e}_i \wedge\wedge * \omega^j e_j, \widehat{e}_0 = e_0\end{aligned}$$

Now

$$-dF^0 \wedge *dX \wedge e_0 = -grad\, F^0 \wedge e_0$$

$$d\bar{F} \wedge\wedge * \omega^0 \wedge e_0 = \frac{\partial \bar{F}}{\partial x^0}dx^0 \wedge \wedge r^2 \sin\theta dr \wedge d\theta \wedge d\phi e_0 = -\frac{\partial \bar{F}}{\partial x^0} dV \wedge e_0$$

$$\begin{aligned}dF^1\widehat{e}_1 \wedge\wedge * \omega^2\widehat{e}_2 &= (dF^r - F^\theta d\theta - F^\phi \sin\theta d\phi)\widehat{e}_1 \wedge r\sin\theta d\phi \wedge dr \wedge dx^0 \wedge \widehat{e}_2 \\ &= (\frac{\partial F^r}{\partial \theta} - F^\theta) d\theta \wedge r\sin\theta d\phi \wedge dr \wedge dx^0 \widehat{e}_1 \wedge \widehat{e}_2 \\ &= \frac{1}{r}(\frac{\partial F^r}{\partial \theta} - F^\theta) dV \widehat{e}_1 \wedge \widehat{e}_2\end{aligned}$$

$$\begin{aligned}dF^2\widehat{e}_2 \wedge\wedge * \omega^1\widehat{e}_1 &= (dF^\theta F^r d\theta - F^\phi \cos\theta d\phi)\widehat{e}_2 \wedge r^2 \sin\theta \wedge d\theta \wedge d\phi \wedge dx^0 \widehat{e}_1 \\ &= \frac{\partial F^\theta}{\partial r} r^2 \sin\theta dr \wedge d\theta \wedge d\phi \wedge dx^0 \widehat{e}_2 \wedge \widehat{e}_1 \\ &= \frac{\partial F^\theta}{\partial r} dV \widehat{e}_2 \wedge \widehat{e}_1\end{aligned}$$

So the coefficient of $\widehat{e}_1 \wedge \widehat{e}_2 = e_r \wedge e_\theta$ is

$$\Big(\frac{1}{r}\frac{\partial F^\theta}{\partial \theta} - \frac{F^\theta}{r} - \frac{\partial F^\theta}{\partial r}\Big) dV = \Big(\frac{1}{r}\frac{\partial F^\theta}{\partial \theta} - \frac{1}{r}\frac{\partial}{\partial r}(rF^\theta)\Big) dV$$

$$\begin{aligned}dF^2\widehat{e}_2 \wedge\wedge * \omega^3\widehat{e}_3 &= (dF^\theta + F^r d\theta - F^\phi \cos\theta d\phi)\widehat{e}_2 \wedge r dr \wedge d\theta \wedge dx^0 \widehat{e}_3 \\ &= \frac{1}{r\sin\theta}(\frac{\partial F^\theta}{\partial \phi} - F^\phi \cos\theta) d\phi \wedge r^2 \sin\theta dr \wedge d\theta \wedge dx^0 \widehat{e}_2 \wedge \widehat{e}_3 \\ &= \frac{1}{r\sin\theta}(\frac{\partial F^\theta}{\partial \phi} - F^\phi \cos\theta) dV \widehat{e}_2 \wedge \widehat{e}_3\end{aligned}$$

$$dF^3\widehat{e}_3 \wedge \wedge * \omega^2\widehat{e}_2 = (F^r \sin\theta d\phi + F^\theta \cos\theta d\phi + dF^\phi)\widehat{e}_3 \wedge r\sin\theta d\phi \wedge dr \wedge dx^0\widehat{e}_2$$
$$= \frac{1}{r}\frac{\partial F^\phi}{\partial\theta}\widehat{e}_3 d\theta \wedge r^2 \sin\theta d\phi \wedge dr \wedge dx^0\widehat{e}_2$$
$$= \frac{1}{r}\frac{\partial F^\phi}{\partial\theta} dV\widehat{e}_3 \wedge \widehat{e}_2$$

The coefficient of $\widehat{e}_2 \wedge \widehat{e}_3 = e_\theta \wedge e_\phi$ is

$$\Big\{\frac{1}{r\sin\theta}(\frac{\partial F^\theta}{\partial\phi} - F^\phi\cos\theta) - \frac{1}{r}\frac{\partial F^\phi}{\partial\theta}\Big\}dV = \frac{1}{r\sin\theta}\Big(\frac{\partial F^\theta}{\partial\phi} - \frac{\partial}{\partial\theta}(F^\phi \sin\theta)\Big)dV$$

$$dF^3\widehat{e}_3 \wedge\wedge * \omega^1\widehat{e}_1 = (F^r \sin\theta d\phi + F^\theta \cos\theta d\phi) + dF^\phi)\widehat{e}_3 \wedge r^2 \sin\theta d\theta \wedge d\phi \wedge dx^0 \wedge \widehat{e}_1$$
$$= (\frac{\partial F^\phi}{\partial r} dr \wedge r^2 \sin\theta \wedge d\theta \wedge d\phi \wedge \wedge dx^0)\widehat{e}_3 \wedge \widehat{e}_1$$
$$= \frac{\partial F^\phi}{\partial r} dV\widehat{e}_3 \wedge \widehat{e}_1$$

$$dF^1\widehat{e}_1 \wedge \wedge * \omega^3\widehat{e}_3 = (dF^r - F^\theta d\theta - F^\phi \sin\theta d\phi)\widehat{e}_1 \wedge r dr \wedge d\theta \wedge dx^0\widehat{e}_3$$
$$= \frac{1}{r\sin\theta}(\frac{\partial F^r}{\partial\phi} - F^\phi \sin\theta)\widehat{e}_1 d\phi \wedge r^2 \sin\theta dr \wedge d\theta \wedge dx^0)\widehat{e}_3$$
$$= \frac{1}{r\sin\theta}(\frac{\partial F^r}{\partial\phi} - F^\phi \sin\theta) dV\widehat{e}_1 \wedge \widehat{e}_3$$

The coefficient of $\widehat{e}_3 \wedge \widehat{e}_1 = e_\phi \wedge e_r$ is

$$\Big\{\frac{\partial F^\phi}{\partial r} - \frac{1}{r\sin\theta}\Big(\frac{\partial F^r}{\partial\phi} - F^\phi \sin\theta\Big)\Big\}dV = \Big(\frac{\partial F^\phi}{\partial r} + \frac{F^\phi}{r} - \frac{1}{r\sin\theta}\frac{\partial F^r}{\partial\phi}]dV$$
$$= \Big[\frac{1}{r} + \frac{\partial}{\partial r}(rF^\phi) - \frac{1}{r\sin\theta}\frac{\partial F^r}{\partial\phi}\Big]dV$$

Taking into account the minus sign we obtain the formula for $curl F$ given in (3.11 d).

Remark 3.1. In N.M.J. Woodhouses book [7] we find invariant expressions for $div\, F$ and $\Box f$ which are obtained by a different procedure. Here, in our approach we have an invariant expression for $curl\, F$ also. We use these expressions to study Maxwells equations and put them in an invariant form.

4. Electromagnetism

In classical electromagnetic field theory the basic Maxwells equations are

$$\text{(a)} \quad div\, D = 4\pi\rho \qquad \text{(b)} \quad div\, B = 0$$
$$\text{(c)} \quad curl\, E = -\frac{\partial B}{\partial x^{0'}}, \quad x^0 = ct \qquad \text{(d)} \quad curl\, H = \frac{4\pi}{c}J + \frac{\partial D}{\partial x^0} \qquad (4.1)$$

where D = dielectric displacement, E = electric field, B =magnetic induction, H =magnetic field, J = electric current density and ρ =charge density. In free space $E = D, B = H, J = 0$ and $\rho = 0$.

We express these equations in forms which are invariant with respect to Lorentz frames, using the theory developed in the previous sections.

Proposition 4.1: The following are invariant with respect to Lorentz transformation equations (1.10).

$$\text{(a) } div\, D = 4\pi\rho \quad \text{(b) } div\, B = 0$$

$$\text{(c) } curl\, E = \frac{\partial}{\partial x^0}(E \wedge e_0 - B) \text{ (d) } curl\, H = \frac{\partial}{\partial x^0}(H \wedge e_0 + D) + \frac{4\pi}{c}J \tag{4.2}$$

Proof: (a) Setting $F = D$ in (3.3 b) and observing that the component D^0 of D in e_0 direction is zero, we have

$$divD = \frac{\partial D^i}{\partial x^i} = 4\pi\rho, \qquad i = 1,2,3 \quad \text{using } (4.1a)$$

The invariance of $divD$ can be established more explicitly in terms of components. We have

$$\begin{aligned} D &= D^\mu e_\mu = D^{\alpha'} e_{\alpha'} \\ dD &= dD^\mu e_\mu = \frac{\partial D^\mu}{\partial x^\gamma} dx^\gamma e_\mu = \frac{\partial D^{\alpha'}}{\partial x^{\beta'}} dx^{\beta'} e_{\alpha'} \end{aligned}$$

But from (1.4) and (1.9) we have

$$e_{\alpha'} = \wedge^\mu_{\alpha'} e_\mu \text{ and } dx^{\beta'} = \wedge^{\beta'}_\gamma dx^\gamma$$

Hence

$$\frac{\partial D^\mu}{\partial x^\gamma} dx^\gamma e_\mu = \frac{\partial Dx^{\alpha'}}{\partial x^{\beta'}} \wedge^{\beta'}_\gamma dx^\gamma \wedge^\mu_{\alpha'} e_\mu$$

which leads to

$$\frac{\partial D^\mu}{\partial x^\gamma} = \frac{\partial D^{\alpha'}}{\partial x^{\beta'}} \wedge^{\beta'}_\gamma \wedge^\mu_{\alpha'}$$

Setting $\mu = \gamma$ we have

$$div\, D = \frac{\partial D^\mu}{\partial x^\mu} = \frac{\partial D^{\alpha'}}{\partial x^{\beta'}} \delta^{\beta'}_{\alpha'} = \frac{\partial D^{\alpha'}}{\partial x^{\alpha'}}$$

(b) The proof is similar to that of (a)

(c) In the expression for $curl\, F$ in (3.5) setting $F^0 = E_0 = 0$ and $\bar{F} = E$ we get

$$curl\, E = \frac{\partial E}{\partial x^0} \wedge_0 + (curl\, E),$$

But by Maxwells equations (4.1) we have $curl\, E = -\frac{\partial B}{\partial x^0}$. This leads to (c).
The proof of (d) is similar to that of (c).

Remark 4.2. The invariant form of equations for $curl\, E$ and $curl\, B$ are different from classical Maxwells equations, the reason being that the components of $curl\, E$ or $curl\, H$ do not follow the order that we have seen for $div\, E$. However relations between the components of $curl\, E$ with respect to Σ and Σ' can be given using equations (1.10). When $\Gamma' = 1$ or equivalently $\nu = 0$, the components become identical. For this purpose we need the following Lemma.

Lemma 4.3. We have

$$\begin{aligned}
1.\quad & dx^{2'}\wedge dx^{3'}\wedge dx^{0'} = [1+n_1^2(\Gamma-1)]dx^2\wedge dx^3\wedge dx^0+(\Gamma-1)n_1n_2dx^3\wedge dx^1\wedge dx^0\\
&\qquad +(\Gamma-1)n_1n_3dx^1\wedge dx^2\wedge dx^0-\Gamma\nu n_1 dx^1\wedge dx^1\wedge dx^2\wedge dx^3\\
2.\quad & dx^{3'}\wedge dx^{1'}\wedge dx^{0'} = [1+n_2^2(\Gamma-1)]dx^3\wedge dx^1\wedge dx^0+(\Gamma-1)n_2n_3dx^1\wedge dx^2\wedge dx^0\\
&\qquad +(\Gamma-1)n_2n_1dx^2\wedge dx^3\wedge dx^0-\Gamma\nu n_2 dx^1\wedge dx^2\wedge dx^3 \qquad (4.3)\\
3.\quad & dx^{1'}\wedge dx^{2'}\wedge dx^{0'} = [1+n_3^2(\Gamma-1)]dx^1\wedge dx^2\wedge dx^0+(\Gamma-1)n_3n_1dx^2\wedge dx^3\wedge dx^0\\
&\qquad +(\Gamma-1)n_3n_2dx^3\wedge dx^1\wedge dx^0-\Gamma\nu n_3 dx^1\wedge dx^2\wedge dx^3\\
4.\quad & dx^{1'}\wedge dx^{2'}\wedge dx^{3'} = -\Gamma\nu(n_1dx^2\wedge dx^3\wedge dx^0+n_2dx^3\wedge dx^1\wedge dx^0+n_3dx^1\wedge dx^2\wedge dx^0)\\
&\qquad +\Gamma dx^1\wedge dx^2\wedge dx^3
\end{aligned}$$

Proof. From (2.1 a) we have

$$dx^{2'}\wedge dx^{3'}\wedge dx^{0'} = \Big\{(A_j\Lambda_k^{0'}-A_k\Lambda_j^{0'})+\Gamma A_j^2{}_k^3 dx^j\wedge dx^k\wedge dx^0+A_j^2{}_k^3\Lambda_l^{0'}dx^j\wedge dx^k\wedge dx^1\Big\}$$

Here

(i) $$A_j\,\Lambda_k^{0'}-A_k\Lambda_j^{0'} = \Gamma^2\nu^2\Big\{-n_3n_k\delta_j^2+n_2n_k\delta_j^3+n_3n_j\delta_k^2-n_2n_j\delta_k^3\Big\}$$

(ii) $$\begin{aligned}\Gamma A_j^2{}_k^3 = \Gamma\{&(\Gamma-1)n_2n_j\delta_k^3+(\Gamma-1)n_3n_k\delta_j^2-(\Gamma-1)n_2n_k\delta_j^3\\ &-(\Gamma-1)n_3n_j\delta_k^2+\delta_k^3\delta_j^2-\delta_k^2\delta_j^3\}\end{aligned}$$

(iii) $$\Lambda_l^{0'}\,A_j^2{}_k^3dx^j\wedge dx^k\wedge dx^l = -\Gamma\nu\Big\{n_1A_2^2{}_3^3+n_2A_3^2{}_1^3+n_3A_1^2{}_3^2\Big\}dx^1dx^2dx^3$$

$$= -\Gamma\nu\Big\{n_1(\Gamma-1)(n_2^2+n_3^2+1)-(\Gamma-1)n_2^2n_1-(\Gamma-1)n_3^2n_1\Big\}dx^1\wedge dx^2\wedge dx^3$$
$$= -\Gamma\nu n_1 dx^1\wedge dx^2\wedge dx^3$$

$$\begin{aligned}
\text{(i)+(ii)} &= \big(n_2n_j\delta_k^3+n_3n_k\delta_j^2\big)\big(\Gamma^2-\Gamma^2\nu^2\big)+\big(n_2n_k\delta_j^3+n_3n_j\delta_k^2\big)\big(\Gamma^2\nu^2-\Gamma^2\big)\\
&\quad +\Gamma(-n_2n_j\delta_k^3-n_3n_k\delta_k^2)+\Gamma(n_2n_k\delta_j^3+n_3n_j\delta_k^2)+\Gamma(\delta_k^3\delta_j^2-\delta_k^2\delta_j^3)\\
&= (1-\Gamma)\big(n_2n_j\delta_k^3+n_3n_k\delta_j^2\big)+(\Gamma-1)\big(n_2n_k\delta_j^3+n_3n_j\delta_k^2\big)+\Gamma\big(\delta_k^3\delta_j^2\delta_k^2\delta_j^3\big)
\end{aligned}$$

Setting $j,k=1,2,3$ in (i)+(ii) above and taking (iii) together we get the expression for $dx^{2'}\wedge dx^{3'}\wedge dx^{0'}$ given in (1).
The proof for 2 and 3 is similar to 1. The proof of 4 is as follows:
From (2.1) we get

$$dx^{1'}\wedge dx^{2'}\wedge dx^{3'}=\left(C_j\Lambda_k^{3'}-C_k\Lambda_j^{3'}+A_j^1{}_k^2\Lambda_0^{3'}\right)dx^j\wedge dx^k\wedge dx^0+A_j^1{}_k^2\Lambda_l^{3'}dx^j\wedge dx^k\wedge dx^l$$

$$=\Big\{\Gamma\nu(n_2\delta_j^1-n_1\delta_j^2)[(\Gamma-1)n_3n_k+\delta_k^3]-\Gamma\nu(n_2\delta_k^1-n_1\delta_k^2)[(\Gamma-1)n_1n_j+\delta_j^3]$$
$$-\Gamma\nu n_j\left[(\Gamma-1)n_1n_j\delta_k^2+(\Gamma-1)n_2n_k\delta_j^1-(\Gamma-1)n_1n_k\delta_j^2-(\Gamma-1)n_2n_j\delta_k^1\right]$$
$$+\delta_j^1\delta_k^2-\delta_k^1\delta_j^2\Big\}dx^j\wedge dx^k\wedge dx^0+\left(A_1^1{}_2^2\Lambda_3^{3'}+A_2^1{}_3^2\Lambda_2^{3'}+A_3^1{}_1^2\Lambda_2^{3'}\right)dx^1\wedge dx^2\wedge dx^0$$

Setting $j,k=1,2,3$ in the first term we get

$$=\Gamma\nu\{-n_1[(\Gamma-1)n_3^2+1]+(\Gamma-1)n_1n_3^2\}dx^2\wedge dx^3\wedge dx^0+\Gamma\nu\{-n_2[(\Gamma-1)n_3^2+1]$$
$$+(\Gamma-1)n_2n_3^2\}dx^3\wedge dx^1\wedge dx^0+\Gamma\nu\{(\Gamma-1)n_3(n_1^2+n_2^2)-(\Gamma-1)n_3(n_1^2+n_2^2)-n_3\}dx^1\wedge dx^2\wedge dx^0$$
$$=-\Gamma\nu n_1dx^2\wedge dx^3\wedge dx^0-\Gamma\nu n_2dx^3\wedge dx^1\wedge dx^0-\Gamma\nu n_3dx^1\wedge dx^2\wedge dx^0$$

$$\text{II term}\quad=\{[(\Gamma-1)(n_1^2+n_2^2)+1][(\Gamma-1)n_3^2+1]-(\Gamma-1)n_1n_3(\Gamma-1)n_1n_3$$
$$+(\Gamma-1)n_3n_2[-(\Gamma-1)n_2n_3]\}dx^1\wedge dx^2\wedge dx^3$$
$$=\{(\Gamma-1)^2n_3^2(n_1^2+n_2^2)+(\Gamma-1)n_3^2+(\Gamma-1)(n_1^2+n_2^2+1)$$
$$+1-(\Gamma-1)^2n_1^2n_3^2-(\Gamma-1)n_2^2n_3^2\}dx^1\wedge dx^2\wedge dx^3$$
$$=\{(\Gamma-1)(n_1^2+n_2^2+n_3^2)+1\}dx^1\wedge dx^2\wedge dx^3=\Gamma dx^1\wedge dx^2\wedge dx^3$$

The first and second term together give the desired result.

Proposition 4.2. The components of $curl\,E$ in Lorentz frames Σ and Σ' are related to each other and their relation are as given below:
For convenience we write $E_1^2{}_2^1=\frac{\partial E_2}{\partial x^1}-\frac{\partial E_1}{\partial x^2}$ and $E_1^2{}_1^2=\frac{\partial E_{2'}}{\partial x^{1'}}-\frac{\partial E_{1'}}{\partial x^{2'}}$ where E_α and $E_{\alpha'}$ are components of electric field E with respect to Σ and Σ'.

$$\begin{aligned}E_1^2{}_2^1&=E_1^2{}_2^1[\Gamma+n_3^2(1-\Gamma)]+E_2^3{}_3^2n_1n_3(1-\Gamma)+E_3^1{}_1^3n_2n_3(1-\Gamma)\\E_2^3{}_3^2&=E_2^3{}_3^2[\Gamma+n_1^2(1-\Gamma)]+E_3^1{}_1^3n_2n_1(1-\Gamma)+E_1^2{}_2^1n_3n_1(1-\Gamma)\\E_3^1{}_1^3&=E_3^1{}_1^3[\Gamma+n_2^2(1-\Gamma)]+E_1^2{}_2^1n_3n_2(1-\Gamma)+E_2^3{}_3^2n_1n_2(1-\Gamma)\end{aligned}\quad(4.4)$$

The two sets of components of $curl\,E$ are identical when $\Gamma=1$ or equivalently when $\nu=0$. It is easy to see that the components of $curlB$ with respect to Σ and Σ' have a similar relation.

Proof. Since

$$E_{\alpha'}dx^{\alpha'}=E_\mu\Lambda_{\alpha'}^{\mu}dx^\gamma\Lambda_\gamma^{\alpha'}=E_\mu dx^\gamma\delta_\gamma^m u=E_\mu dx^\mu$$

we write

$$\begin{aligned} M = E_\mu dx^\mu \wedge dx^0 &= E_{\alpha'} dx^{\alpha'} \wedge dx^{\beta'} \wedge^0_{\beta'} \\ &= \Gamma E_{\alpha'} dx^{\alpha'} \wedge dx^{0'} + E_{\alpha'} dx^{\alpha'} \wedge dx^{j'} \wedge^0_{j'} \end{aligned}$$

$$dM = \frac{\partial E_\mu}{\partial x^\gamma} dx^\gamma \wedge dx^\mu \wedge dx^0 = \Gamma \frac{\partial E_{\alpha'}}{\partial x^{\beta'}} dx^{\beta'} \wedge dx^{\alpha'} \wedge dx^{0'} + \frac{\partial E_{\alpha'}}{\partial x^{\beta'}} dx^{\beta'} \wedge dx^{\alpha'} \wedge dx^{j'} \wedge^0_{j'}$$

Since $E_0 = 0$ and $dx^0 \wedge dx^0 = 0$, we write

$$dM = \frac{\partial E_j}{\partial x^j} dx^j \wedge dx^j \wedge dx^0 = \Gamma \frac{\partial E_{j'}}{\partial x^{j'}} dx^{j'} \wedge dx^{j'} \wedge dx^{0'} + \wedge^0_{l'} \frac{\partial E_{j'}}{\partial x^{j'}} dx^{j'} \wedge dx^{j'} \wedge dx^{l'}$$

Thus

$$dM = E^{2}_{1}{}^{1}_{2} dx^1 \wedge dx^2 \wedge dx^0 + E^{3}_{2}{}^{2}_{3} dx^2 \wedge dx^3 \wedge dx^0 + E^{1}_{3}{}^{3}_{1} dx^3 \wedge dx^1 \wedge dx^0 \quad (4.5)$$

Also

$$dM = \Gamma\big(E'^{2}_{1}{}^{1}_{2} dx^{1'} \wedge dx^{2'} \wedge dx^{0'} + E'^{3}_{2}{}^{2}_{3} dx^{2'} \wedge dx^{3'} \wedge dx^{0'} + E'^{1}_{3}{}^{3}_{1} dx^{3'} \wedge dx^{1'} \wedge dx^{0'}\big)$$

$$+\Gamma \nu Q dx^{1'} \wedge dx^{2'} \wedge dx^{3'} \quad (4.6)$$

where

$$Q = \big(n_3 E'^{2}_{1}{}^{1}_{2} + n_1 E'^{3}_{2}{}^{2}_{3} + n_2 E'^{1}_{3}{}^{3}_{1}\big) \quad (4.7)$$

Using Lemma (4.3), we get

$$\begin{aligned} dM =\ & \Gamma\big[E'^{2}_{1}{}^{1}_{2}(1+n_3^2(\Gamma-1)) + E'^{3}_{2}{}^{2}_{3}(\Gamma-1)n_1n_3 + E'^{1}_{3}{}^{3}_{1}(\Gamma-1)n_2n_3\big] dx^1 \wedge dx^2 \wedge dx^0 \\ & + \big[E'^{3}_{2}{}^{2}_{3}(1+n_1^2(\Gamma-1)) + E'^{1}_{3}{}^{3}_{1}(\Gamma-1)n_1n_2 + E'^{2}_{1}{}^{1}_{2}(\Gamma-1)n_3n_1\big] dx^2 \wedge dx^3 \wedge dx^0 \\ & + \big[E'^{1}_{3}{}^{3}_{1}(1+n_2^2(\Gamma-1)) + E'^{3}_{2}{}^{2}_{3}(\Gamma-1)n_1n_2 + E'^{2}_{1}{}^{1}_{2}(\Gamma-1)n_2n_3\big] dx^3 \wedge dx^1 \wedge dx^0 \\ & -\Gamma^2 \nu Q dx^1 \wedge dx^2 \wedge dx^3 \\ & + \Gamma\nu Q\big[(-\Gamma\nu n_1 dx^2 \wedge dx^3 \wedge dx^0 - \Gamma\nu n_2 dx^3 \wedge dx^1 \wedge dx^0 - \Gamma\nu n_3 dx^1 \wedge dx^2 \wedge dx^0) + \Gamma dx^1 \wedge dx^2 \wedge dx^3\big] \\ =\ & \Gamma\big\{\big[E'^{2}_{1}{}^{1}_{2}(1+n_3^2(\Gamma-1)) + E'^{3}_{2}{}^{2}_{3}(\Gamma-1)n_1n_3 + E'^{1}_{3}{}^{3}_{1}(\Gamma-1)n_1n_2 - \Gamma^2\nu^2 n_3 Q\big] dx^1 \wedge dx^2 \wedge dx^0 \\ & + \big[E'^{3}_{2}{}^{2}_{3}(1+n_1^2(\Gamma-1)) + E'^{1}_{3}{}^{3}_{1}(\Gamma-1)n_1n_2 + E'^{2}_{1}{}^{1}_{2}(\Gamma-1)n_1n_3 - \Gamma^2\nu^2 n_1 Q\big] dx^2 \wedge dx^3 \wedge dx^0 \\ & + \big[E'^{1}_{3}{}^{3}_{1}(1+n_2^2(\Gamma-1)) + E'^{3}_{2}{}^{2}_{3}(\Gamma-1)n_1n_2 + E'^{2}_{1}{}^{1}_{2}(\Gamma-1)n_2n_3 - \Gamma^2\nu^2 n_2 Q\big]\big\} dx^3 \wedge dx^1 \wedge dx^0 \end{aligned} \quad (4.8)$$

Consider the first term

$$E'^{2}_{1}{}^{1}_{2}[\Gamma + n_3^2(\Gamma^2 - \Gamma)] + E'^{3}_{2}{}^{2}_{3}(\Gamma^2 - \Gamma)n_1n_3 + E'^{1}_{3}{}^{3}_{1}(\Gamma^2 - \Gamma)n_2n_3$$

$$-\Gamma^2\nu^2\big(n_3^2 E'^{2}_{1}{}^{1}_{2} + n_1n_3 E'^{2}_{1}{}^{1}_{2} + n_1n_3 E^{3}_{2}{}^{2}_{3} + n_2n_3 E'^{1}_{3}{}^{3}_{1}\big)$$

$$= E'^{2}_{1}{}^{1}_{2}[\Gamma - n_3^2\Gamma + n_3^2(\Gamma^2 - \Gamma\nu^2)] + E'^{3}_{2}{}^{2}_{3}n_1n_3(\Gamma^2 - \Gamma\nu^2 - \Gamma) + E'^{1}_{3}{}^{3}_{1}n_2n_3(\Gamma^2 - \Gamma - \Gamma^2\nu^2)$$

$$= E'^{2}_{1}{}^{1}_{2}[\Gamma + n_3^2(1-\Gamma)] + E'^{3}_{2}{}^{2}_{3}n_1n_3(1-\Gamma) + E'^{1}_{3}{}^{3}_{1}n_2n_3(1-\Gamma)$$

Second and third terms have similar expressions.
Finally comparing the coefficient of $dx^1 \wedge dx^2 \wedge dx^0$ etc., (4.6) and the simplified version of (4.8) we get the results in (4.4).

In the expression for $curl\, E$, we have the term $\frac{\partial E}{\partial x^0} \wedge l_0$. The relation between the components of this term with respect to Σ and Σ' is as follows:

$$\frac{\partial E}{\partial x^\alpha}dx^\alpha = dE = \frac{\partial E}{\partial x^{\beta'}}dx^{\beta'}$$

$$\begin{aligned}\frac{\partial E}{\partial x^0}dx^0 + \frac{\partial E}{\partial x^i}dx^i &= \frac{\partial E}{\partial x^{0'}}dx^{0'} + \frac{\partial E}{\partial x^{i'}}dx^{i'} \\ &= \frac{\partial E}{\partial x^{0'}} \wedge^{0'}_{\alpha} dx^\alpha + \frac{\partial E}{\partial x^{i'}} \wedge^{i'}_{i} dx^i \\ &= \frac{\partial E}{\partial x^{0'}} \wedge^{0'}_{0'} dx^0 + \frac{\partial E}{\partial x^{0'}} \wedge^{0'}_{i} dx^i + \frac{\partial E}{\partial x^{i'}} \wedge^{i'}_{i} dx^i\end{aligned}$$

Since dx^0, $dx^i, \cdots$ etc, are linearly independent. We have

$$\frac{\partial E}{\partial x^0} = \Gamma \frac{\partial E}{\partial x^{0'}}$$

$$\frac{\partial E^i}{\partial x^0}e_i = \Gamma \frac{\partial E^{j'}}{\partial x^{0'}} \wedge^{i}_{j'} e_j$$

This leads to

$$\frac{\partial E^i}{\partial x^0} = \Gamma\big[(\Gamma - 1)n_in_j + \delta^i_j\big]\frac{\partial E^{j'}}{\partial x^{0'}}$$

When $\Gamma = 1$, we have

$$\frac{\partial E^{i'}}{\partial x^{0'}} = \frac{\partial E^i}{\partial x^0}.$$

A significant special case of (4.4) is the following:

Suppose Σ' moves with uniform velocity v along one of the coordinate axes x^1, x^2, x^3 of Σ, say for instance, along x^3, then $n_3 = 1$, $n_1 = 0$, $n_2 = 0$ and (4.4) gives

$$E^{2}_{1}{}^{1}_{2} = E'^{2}_{1}{}^{1}_{2}, \quad E^{3}_{2}{}^{2}_{3} = \Gamma E'^{3}_{2}{}^{2}_{3}, \quad E^{1}_{3}{}^{3}_{1} = \Gamma E'^{1}_{3}{}^{3}_{1}$$

In other words $E^{3}_{2}{}^{2}_{3}$ and $E^{1}_{3}{}^{3}_{1}$ undergo dilation, while $E^{3}_{1}{}^{1}_{2}$ remains unaffected.
Remark 4.2 There is a well known method [2][3] which uses differential forms to derive Maxwell's equations in their usual form. We briefly mention the same here.

Let

$$\alpha = \sum_{i=1}^{3} E_i dx^i \wedge dx^0 + \left(B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2\right)$$

$$d\alpha = \sum_{i=1}^{3} \frac{\partial E_i}{dx^j} dx^j \wedge dx^i \wedge dx^0 + \Big[\frac{\partial B_1}{\partial x^0} dx^0 \wedge dx^2 \wedge dx^3 + \frac{\partial B_2}{\partial x^0} dx^0 \wedge dx^3 \wedge dx^1$$
$$+ \frac{\partial B_3}{\partial x^0} dx^0 \wedge dx^1 \wedge dx^2 + div\, B\, dx^1 \wedge dx^2 \wedge dx^3\Big]$$
$$= \left(E^{3\ 2}_{2\ 3} + \frac{\partial B_1}{\partial x^0}\right) dx^2 \wedge dx^3 \wedge dx^0 + \left(E^{1\ 3}_{3\ 1} + \frac{\partial B_2}{\partial x^0}\right) dx^3 \wedge dx^1 \wedge dx^0$$
$$+ \left(E^{2\ 1}_{1\ 2} + \frac{\partial B_3}{\partial x^0}\right) dx^1 \wedge dx^2 \wedge dx^0 + (div\, B) dx^1 \wedge dx^2 \wedge dx^3$$

where $E^{3\ 2}_{2\ 3} = \frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3}$, etc

Setting $d\alpha = 0$ and observing that $dx^2 \wedge dx^3 \wedge dx^0$ etc are linearly independent we get (4.1) (b) and (c) in terms of components.

Further let

$$\beta = -\Big(\sum_{j=1}^{3} H_i dx^i \wedge dx^0\Big) + D_1 dx^2 \wedge dx^3 + D_2 dx^3 \wedge dx^1 + D_3 dx^1 \wedge dx^2$$

$$\gamma = -\left(J_1 dx^2 \wedge dx^3 + J_2 dx^3 \wedge dx^1 + J_3 dx^1 \wedge dx^2\right) - \rho dx^1 \wedge dx^2 \wedge dx^3$$

Then $d\beta + 4\pi\gamma = 0$ gives (4.1 a) and (d).

5. Null Hypersurfaces of a Minkowski space

Let p be a point of (M, η). A non-zero vector at p is said to be timelike, null or spacelike according to whether $\eta(V, V)$ is negative, zero or positive respectively. Let H be a a set of points determined by x^1, x^2, x^3 and $x^0 = f(x^1, x^2, x^3)$ where f is real valued function. H is called hypersurface of (M, η) and the metric on M induces a metric on H.

A point X of H is given by $X = x^i e_j + f(x^1, x^2, x^3) e_0$, then

$$dX = dx^j e_j + p_i dx^i e_0 = dx^i \widehat{e}_i$$

where $p_i = \dfrac{\partial f}{\partial x^i}$ and $\widehat{e}_i = e_i + p_i e_0$

A vector V at p is null in (M, η) if

$$0 = \eta_{\alpha\beta} V^\alpha V^\beta = (V^1)^2 + (V^2)^2 + (V^3)^2 - (V^0)^2 \qquad (5.1)$$

If V is null with respect to hypersurface H also, then $\langle V, \tilde{e}_i \rangle = 0 \;\; i = 1, 2, 3$, that is

$$\langle V^i e_i + V_0 e_0,\ e_i + p_i e_0 \rangle = V^i - p_i V^0 = 0$$

Substituting for V^i in terms of $p_i V^0$ in (5.1) we have $(V^0)^2(p_1^2 + p_2^2 + p_3^2 - 1) = 0$. If $V^0 \neq 0$, then $\Sigma p_j^2 - 1 = 0$. This indeed is the condition for H to be a null hypersurface.

Example 5.1 The null cone: Let

$$x^0 = f(x^1, x^2, x^3) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$$

Then we have

$$p_1 = \frac{x^1}{x^0}, \quad p_2 = \frac{x^2}{x^0}, \quad p_3 = \frac{x^3}{x^0}$$

So that $\sum_{i=1}^{3} p_i^2 - 1 = 0$. Hence the cone $(x_0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ is a null cone.

This example is well known [4] but we have used it to illustrate the general theory developed above.

Example 5.2 The null Hyperplane: Let

$$f(x^1, x^2, x^3) = \frac{a_1 x^1 + a_2 x^2 + a_3 x^3}{\sqrt{\sum_{i=1}^{3} a_i^2}}$$

Here $p_i = \frac{a_i}{\sqrt{\sum_{i=1}^{3} a_i^2}}$, $i = 1, 2, , 3$. Hence $\sum_{i=1}^{3} p_i^2 = 1$ and so

$$x^0 = \frac{a_1 x^1 + a_2 x^2 + a_3 x^3}{\sqrt{\sum_{i=1}^{3} a_i^2}} \tag{5.7}$$

where all a_i's do not vanish simultaneously, is a null hyperplane.

Proposition 5.1 The Null hyperplane is a tangent plane to the null cone.

Proof. We show that the null hyperplane cuts the null cone in two coincident lines. Consider

$$x^0 = (x^1)^2 + (x^2)^2 + (x^3)^2 = \frac{a_1 x^1 + a_2 x^2 + a_3 x^3}{a_1^2 + a_2^2 + a_3^2}$$

Cross multiplying and simplifying we get

$$(a_2 x^1 - a_1 x^2)^2 + (a_3 x^2 - a_3 x^2)^2 + (a_1 x^3 - a_3 x^1)^2 = 0$$

from which we get the common line to both the null cone and the hyperplane, namely

$$\frac{x^1}{a_1} = \frac{x^2}{a_2} = \frac{x^3}{a_3} = \pm \frac{x^0}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

This gives the required result.

6. Spacelike Hypersurfaces of a Minkowski space
It is a well known result [4] that the hypersurface $H' : (x_0)^2 - (x^1)^2 - (x_2)^2 - (x_3)^2 = 1$ is a space like hypersurface and the timelkne geodesics through the origin 0 are orthogonal to Σ. We give below a simple proof of this result.

Let P be a point of H, then

$$OP = \sum_{i=1}^{3} x^i e_i + \sqrt{1 + \sum_{i=1}^{3} (x^i)^2 e_0} \tag{6.1}$$

We write $OP = X$, and the observe that X is a timelkne vector and

$$dX = \sum_{i=1}^{3} dx^i e_i + \frac{\sum_{i=1}^{3} x^i dx^i}{\sqrt{1 + \sum_{i=1}^{3} (x^i)^2}} e_0 = \sum_{i=1}^{3} dx^i \widehat{e}_i \text{ say} \tag{6.2}$$

where

$$\widehat{e}_i = e_i + \frac{x^i}{\sqrt{1 + \sum_{i=1}^{3} (x^i)^2}} e_0 \quad i = 1, 2, 3 \tag{6.3}$$

Notice that $\eta(\widehat{e}_i, \widehat{e}_j) = 1 - \frac{(x^i)^2}{\sqrt{1 + \sum_{i=1}^{3} (x^i)^2}} > 0$

which shows that each $\widehat{e}_i$ is spacelike. This justifies calling H', a spacelike hypersurface.

Proposition 6.1 OP is a orthogonal to the spacelike hyper-surface H'.

Proof: It is enough if we show that $\eta(OP, \tilde{e}_i) = 0$ for all $i = 1, 2, 3$. We have

$$\eta(OP, \widehat{e}_i) = x^i - \sqrt{1 + \sum_{i=1}^{3} (x^i)^2} - \frac{x^i}{\sqrt{1 + \sum_{i=1}^{3} (x^i)^2}} = 0, \quad i = 1, 2, 3$$

This establishes the orthogonality of the timeline vector OP with the hypersurface Σ.

Now consider the hyper-surface H' defined by

$$-(x^0)^2 + \sum_{i=1}^{3} (x^i)^2 = 1$$

Here

$$OP = \sum_{i=1}^{3} x^i e_i + \sqrt{\sum_{i=1}^{3}(x^i)^2 - 1}\; e_0$$

It is easy to see that OP is a spacelike vector. Writing $OP = X$ we have

$$dX = \sum_{i=1}^{3} dx^i e_i + \frac{\sum_{i=1}^{3} x^i dx^i}{\sqrt{\sum_{i=1}^{3}(x^i)^2 - 1}} e_0 = \sum_{i=1}^{3} dx^i \widehat{e}_i \quad \text{say}$$

$$\widehat{e}_i = e_i + \frac{x^i}{\sqrt{\sum_{i=1}^{3}(x^i)^2 - 1}} e_0$$

As in the case of H, we have

$$\eta(OP, \widehat{e}_i) = 0 \;\text{ for all }\; i = 1, 2, 3$$

Appendix: The Star Operator

Let V be a vectorspace over R with an inner product $\langle\, ,\, \rangle$ and an orthogonal basis $\sigma^1, \cdots, \sigma^n$. Let

$$\sigma^H = \sigma^1 \wedge \cdots \wedge \sigma^p \;\text{ and }\; \sigma^K = \sigma^{p+1} \wedge \cdots \wedge \sigma^n$$

It is known [2] that

$$*\sigma^H = \langle \sigma^K, \sigma^K \rangle \sigma^K$$

where

$$\langle \sigma^K, \sigma^K \rangle = \langle \sigma^{P+1}, \sigma^{P+1} \rangle \cdots \langle \sigma^n, \sigma^n \rangle$$

Now let $V = T_p^*(M)$ where M is a Minkowski space of dimension 4. An orthonormal basis of $T_p^*(M)$ is dx^0, dx^1, dx^2, dx^3. The inner product in $T_p^*(M)$ is given by

$$\langle dx^\alpha, dx^\beta \rangle = \eta^{\alpha\beta}.$$

Consider

$$\begin{aligned} *dx^0 &= \langle dx^1, dx^1 \rangle \langle dx^2, dx^2 \rangle \langle dx^3, dx^3 \rangle dx^1 \wedge dx^2 \wedge dx^3 \\ &= dx^1 \wedge dx^2 \wedge dx^3 \end{aligned}$$

Here $H = \{0\}$ and $K = \{1, 2, 3\}$

Writing $dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = -dx^1 \wedge dx^0 \wedge dx^2 \wedge dx^3$ and taking $H = \{1\}$ and $K = \{0, 2, 3\}$ we get

$$\begin{aligned} *dx^1 &= \langle dx^0, dx^0 \rangle \langle dx^2, dx^2 \rangle \langle dx^3, dx^3 \rangle (-dx^0 \wedge dx^2 \wedge dx^3) \\ &= dx^0 \wedge dx^2 \wedge dx^3, \text{ since } \langle dx^0, dx^0 \rangle = -1 \\ &= dx^2 \wedge dx^3 \wedge dx^0 \end{aligned}$$

Similarly we get $*dx^2 = dx^3 \wedge dx^1 \wedge dx^0$ and $*dx^3 = dx^1 \wedge dx^2 \wedge dx^0$.
With these values of $*dx^0$, $*dx^1$, $*dx^2$, and $*dx^3$ it is easy to observe that

$$dx^\alpha \wedge *dx^\beta = \eta^{\alpha\beta} dV$$

where $dV = dx^1 \wedge dx^3 \wedge dx^3 \wedge dx^0$.

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