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Subspaces of the Generalized Lagrange Space with the Metric $g_{ij}(x,y) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x)}\right) y_i y_j$

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1. Introduction

R. Miron and M. Anastesiei [4] have developed theory of subspaces of generalized Lagrange spaces to a large extent in their monograph "Vector bundles and Lagrange spaces, application in relativity". In 1989 T. Kawaguchi and R. Miron [3] gave a class of generalized Lagrange space $M^n = (M, g_{ij}(x,y))$ where

$$g_{ij}(x,y) = \gamma_{ij}(x) + \frac{1}{c^2} y_i y_j,$$

$\gamma_{ij}(x)$ being a Riemannian metric on the $n-$dimensional differentiable manifold M and $y_i = \gamma_{ij}(x)\, y^j$. J. L. Synge [7], M. C. Chaki and B. Barua [2] used the metric

$$g_{ij}(x, v(x)) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x, v(x))}\right) v_i v_j,$$

which occurs in the study of relativistic optics [6]. In this case (x) is a generic point, $v(x)$ is the velocity vector of the point and $\eta(x, v(x))$ is the refractive index of the optical medium. If, in particular $\eta(x, v(x)) = 1$, the medium is transparent. Also, if the refractive index is independent of velocity, i.e. if $\eta = \eta(x)$ then the optical medium is called non dispersive.

In this paper we use the metric $g_{ij}(x,y) = \gamma_{ij}(x) + \left(1 - \dfrac{1}{\eta^2(x)}\right) y_i y_j$ and we denote the generalized Lagrange space with this metric as GL^n. The purpose of the present paper is to discuss the properties of subspace of GL^n.

2. Generalized Lagrange space GL^n

Let M be a $n-$dimensional differentiable manifold, (TM, π, M) its tangent bundle and (x^i, y^i) $(i, j, ... = 1, 2, 3, ..., n = \dim M)$ the canonical coordinates of the points $u \in TM$, $\pi(u) = x$ in a coordinate neighbourhood $\pi^{-1}(U)$, where U is a coordinate neighbourhood of M at x i. e. $x \in U \subset M$. One can consider a Riemannian metric $\gamma_{ij}(x)$ on M and the Riemannian space $V^n = (M, \gamma_{ij}(x))$.

The Liouville vector field $y = y^i \dfrac{\partial}{\partial y^i}$ is globally defined on the total space TM. Thus the covector field

$$y^i = \gamma_{ij}(x)\, y^j \tag{2.1}$$

is globally defined on TM, and also the square of the norm of y and the functions a_σ is defined respectively by

$$||y||^2 = \gamma_{ij}\,(x) y^i\, y^j, \tag{2.2}$$

$$a_\sigma(x, y) = 1 + \sigma \left[1 - \frac{1}{\eta^2(x)}\right] ||y||^2,\ \sigma \in N \text{ (the set of natural numbers)}. \tag{2.3}$$

On TM we can consider the $d-$tensor field

$$g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{\eta^2(x)}\right) y_i y_j. \tag{2.4}$$

The reciprocal of tensor field $g^{ij}(x, y)$ is given by

$$g^{ij}(x, y) = \gamma^{ij}(x) - \frac{1}{a_1(x, y)} \left(1 - \frac{1}{\eta^2(x)}\right) y^i y^j. \tag{2.5}$$

The $d-$tensor field C_{jhk} is defined by

$$C_{jhk} = g_{hr}\, C^r_{jk} = \frac{1}{2}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right).$$

Using (2.4) we get

$$C_{jhk} = \left(1 - \frac{1}{\eta^2(x)}\right) \gamma_{jk} y_h.$$

Then from (2.5) we get

$$C^i_{jk} = \frac{1}{a_1}\left(1 - \frac{1}{\eta^2(x)}\right)\gamma_{jk} y^i. \tag{2.6}$$

The non-linear connection of the space GL^n is given by [3]

$$N^i_j(x,y) = \{{}_j{}^i{}_k\}\, y^k, \tag{2.7}$$

where $\{{}_j{}^i{}_k\}$ is the Christoffel symbol of the Riemannian space V^n constructed from $\gamma_{ij}(x)$. The canonical metrical connection L^i_{jk} of the space GL^n is defined by

$$L^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right)$$

where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$. Using equations (2.4) and (2.7) we get $L^i_{jk} = \{()_j{}^i{}_k\}$.

3. Subspace of generalized Lagrange space

Let $\overline{M}$ be a differentiable manifold of dimension $m, 1 \leq m < n$ immersed in the $n-$dimensional manifold M by the immersion $i : \overline{M} \to M$. Locally the immersion i can be given in the form

$$x^i = x^i\,(u^1, u^2, u^3, \ldots, u^m), \qquad \text{rank}\left\|\frac{\partial x^i}{\partial u^\alpha}\right\| = m. \tag{3.1}$$

Throughout this paper the indices $i, j, k, ...$ take the values $1, 2, 3, ..., n$ and the indices $\alpha, \beta, \gamma, ...$ take the values $1, 2, 3, ..., m$.

In this case when i is embedding, we shall identify $\overline{M}$ with $i(\overline{M})$ and we shall say that $\overline{M}$ is a submanifold of the manifold M. The equations (3.1) will be called the parametric equations of the submanifold $\overline{M}$ of M.

The derivatives

$$B^i_\alpha(u) = \frac{\partial x^i}{\partial u^\alpha},\ (\alpha = 1, 2, 3, ..., m), \tag{3.2}$$

determine m local vector field on $\overline{M}$. The immersion $i : \overline{M} \to M$ induces an immersion $i^* : T\overline{M} \to M$. The point $(u, v) \in T\overline{M}$ is applied by i^* in the point $(x(u), y(u))$. We have

$$y^i = B^i_\alpha(u)\, v^\alpha, \tag{3.3}$$

and we put

$$(3.4)\qquad B^i_{\alpha\beta}(u)=\frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta},\qquad B^i_{0\beta}(u)=v^\alpha B^i_{\alpha\beta}.$$

The generalized Lagrange metric $g_{ij}(x,y)$ of GL^n induces a metric on $T\overline{M}$: a symmetric and positively defined $d-$tensor field $g_{\alpha\beta}$ is given by

$$(3.5)\qquad g_{\alpha\beta}=g_{ij}(x(u),y(u))\,B^i_\alpha(u)\,B^j_\beta(u).$$

The pair $G\bar{L}^m=(\overline{M},g_{\alpha\beta}(u,v))$ is called the generalized Lagrange subspace of the generalized Lagrange space GL^n. We have a set of $n-m$ unit normal vectors B_λ to the subspace $G\bar{L}^m$ defined by

$$(3.6)(a)\qquad g_{ij}B^i_\lambda B^j_\mu=\delta_{\lambda\mu},\quad (b)\quad g_{ij}B^i_\alpha B^j_\lambda=0,\qquad \lambda,\mu=1,2,3,...,(n-m).$$

The inverse of the matrix of $\left|\left|B^i_\alpha(u)\quad B^i_\lambda(u,v)\right|\right|$ will be denoted by $||B^\alpha_i(u,v)$ $B^\lambda_i(u,v)\big|\big|$. Thus we have

$$(3.7)\qquad B^\alpha_i B^i_\beta=\delta^\alpha_\beta,\qquad B^\lambda_i B^i_\alpha=0,\qquad B^\alpha_i B^i_\lambda=0$$

$B^\lambda_i B^i_\mu=\delta^\lambda_\mu$ and $B^i_\alpha B^\alpha_j+B^i_\lambda B^\lambda_j=\delta^i_j.$

From (3.5) and (3.6) we deduce

$$(3.8)\qquad g_{\alpha\beta}B^\alpha_i=g_{ij}B^j_\beta,\qquad \delta_{\lambda\mu}B^\mu_i=g_{ij}B^j_\lambda.$$

The non-linear connection $\underline{N}\,(\underline{N}^\alpha_\beta(u,v))$ on $T\overline{M}$ induced by the non-linear connection $N(N^i_j(x,y))$ is given by [4]

$$(3.9)\qquad \underline{N}^\alpha_\beta(u,v)=B^\alpha_i(u,v)\{B^i_{0\beta}(u,v)+N^i_j(x(u),y(u,v))B^j_\beta(u)\}.$$

Let $N(N^i_j)$ and $\underline{N}(\underline{N}^\alpha_\beta)$ be non-linear connections on TM and $T\overline{M}$ respectively, then the connection $\underline{D}\Gamma(\underline{N})=(L^\alpha_{\beta\gamma}(u,v),\,C^\alpha_{\beta\gamma}(u,v))$ induced by the metrical $d-$connection $D\Gamma(N)=(L^i_{jk}(x,y),\,C^i_{jk}(x,y))$ is given by [4]

$$(3.10)(a)\qquad L^\alpha_{\beta\gamma}=B^\alpha_i(B^i_{\beta\gamma}+B^j_\beta L^i_{j\gamma}),$$

$$(b)\qquad C^\alpha_{\beta\gamma}=B^\alpha_i B^j_\beta C^i_{j\gamma},$$

where

$$(3.11)(a)\qquad L^i_{j\alpha}=B^k_\alpha L^i_{jk}+H^\lambda_\alpha B^k_\lambda C^i_{jk},$$

$$(b)\qquad C^i_{j\alpha}=B^k_\alpha C^i_{jk},$$

$$(c)\qquad H^\lambda_\alpha=B^\lambda_i(B^i_{0\alpha}+N^i_j B^j_\alpha).$$

Hence $H^{\lambda}_{\alpha}(u,v)$ are components of a mixed $d-$tensor field which has been called first fundamental $h-$tensor in the case of Finsler space [4].

The induced canonical metrical $d-$connection $\underline{D}\Gamma(\underline{N})$ of $G\overline{L}^m$ has the following properties:

$$\text{(3.12)(a)} \qquad g_{\alpha\beta|\gamma}=0, \quad g_{\alpha\beta}\,|_{\gamma}\,=0$$

$$\text{(b)} \qquad v^{\alpha}{}_{|\beta}=0, \quad ||v||^2{}_{|\beta}=0, \quad ||v||^2\Big|_{\beta}=2v_{\beta},$$

$$\text{(c)} \qquad \gamma_{\alpha\beta|\gamma}=0, \quad \gamma_{\alpha\beta}\,|_{\gamma}\,=-\frac{1}{a_1}\left(1-\frac{1}{\eta^2(x)}\right)(\gamma_{\alpha\beta}v_{\gamma}+\gamma_{\beta\gamma}v_{\alpha}),$$

$$\text{(d)} \qquad D^{\alpha}_{\beta}=N^{\alpha}_{\beta}-L^{\alpha}_{\beta\gamma}v^{\gamma}=0, \quad v^{\alpha}\,|_{\beta}\,=\delta^{\alpha}_{\beta}-\tfrac{1}{a_1}\left(1-\tfrac{1}{\eta^2(x)}\right)v^{\alpha}v_{\beta},$$

$$\text{(e)} \qquad C^{\alpha}_{\beta\gamma|\delta}=0, \quad C^{\alpha}_{\beta\gamma}\,|_{\delta}\,=\frac{1}{a_1}\left(1-\frac{1}{\eta^2(x)}\right)\left\{\delta^{\alpha}_{\delta}\gamma_{\beta\gamma}-\frac{v^{\alpha}}{a_1}\left(1-\frac{1}{\eta^2(x)}\right)\right.$$
$$\left.(\gamma_{\beta\delta}v_{\gamma}+\gamma_{\gamma\delta}v_{\beta}-\gamma_{\beta\gamma}v_{\delta})\right\}$$

The $h-$ and $v-$covariant derivatives of B^i_{α} are given by

$$B^i_{\alpha|\beta}=H^{\lambda}_{\alpha\beta}B^i_{\lambda}, \qquad B^i_{\alpha}\Big|_{\beta}=K^{\lambda}_{\alpha\beta}B^i_{\lambda}\ ,$$

where

$$\text{(3.13)(a)} \qquad H^{\lambda}_{\alpha\beta}=B^{\lambda}_i\,(B^i_{\alpha\beta}+B^j_{\alpha}L^i_{j\beta}),$$

$$\text{(b)} \qquad K^{\lambda}_{\alpha\beta}=B^{\lambda}_iB^j_{\alpha}C^i_{j\beta}).$$

The quantities $H^{\lambda}_{\alpha\beta}$ and $K^{\lambda}_{\alpha\beta}$ are components of mixed tensor fields. These tensor fields have been called the second fundamental h-and v-tensor fields respectively in the case of Finsler space [4].

4. Subspace of GLn with metric $g_{ij}(x,y)=\gamma_{ij}(x)+\left(1-\frac{1}{\eta^2(x)}\right)y_iy_j$

Consider a Riemannian subspace V^m of Riemannian space $V^n=(M,\gamma_{ij}(x))$ and subspace $G\overline{L}^m$ of the generalized Lagrange space $GL^n=(M,g_{ij}(x,y))$ given by (2.4). Let n^i_{λ} be the set of $(n-m)$ unit normal vectors to the subspace V^m of V^n defined by

(4.1) $\quad \gamma_{ij}\,n^i_{\lambda}n^j_{\mu}=\delta_{\lambda\mu}, \qquad \gamma_{ij}B^i_{\alpha}n^j_{\lambda}=0, \qquad \lambda,\mu=1,2,3,...,(n-m)$ and let $(B^{\alpha}_i\ n^{\lambda}_i)$ be the inverse matrix of $(B^i_{\alpha}\ n^i_{\lambda})$.

The functions B^i_α may be regarded as components of m linearly independent tangent vectors of both the subspaces V^m and $G\overline{L}^m$. In view of equation (3.3) and (4.1), we have

$$(4.2) \qquad y_i\, n^i_\lambda = 0.$$

Thus the equations (2.4) and (4.1) give

$$(4.3) \qquad g_{ij}\, n^i_\lambda n^j_\mu = \delta_{\lambda\mu}, \qquad g_{ij}\, B^i_\alpha n^j_\lambda = 0.$$

Hence we have the following:

Theorem (4.1). The linear frame $(B^i_1, B^i_2, ..., B^i_m, n^i_1, n^i_2, ..., n^i_{n-m})$ of V^n is same the linear frame of GL^n such that either (4.1) is satisfied along V^m or (3.6) is satisfied along $G\overline{L}^m$. In particular $B^i_\lambda = n^i_\lambda$ for $\lambda = 1, 2, 3, ..., (n-m)$.

From equations (2.4), (3.3) and (3.5) it follows that

$$(4.4) \qquad g_{\alpha\beta}(u, v) = \gamma_{\alpha\beta}(u) + \left(1 - \frac{1}{\eta^2(x)}\right) v_\alpha v_\beta,$$

where

$$(4.5) \qquad \gamma_{\alpha\beta} = \gamma_{ij}(x)\, B^i_\alpha B^j_\beta, \qquad v_\alpha = \gamma_{\alpha\beta} v^\beta = y_i\, B^i_\alpha.$$

The reciprocal $d-$tensor field $g^{\alpha\beta}(u, v)$ of $g_{\alpha\beta}(u, v)$ is given by

$$(4.6) \qquad g^{\alpha\beta}(u, v) = \gamma^{\alpha\beta}(u) - \frac{1}{a_1}\left(1 - \frac{1}{\eta^2(x)}\right) v^\alpha v^\beta.$$

From (3.3) and (4.5), we have

$$||y||^2 = \gamma_{ij}(x) y^i y^j = \gamma_{\alpha\beta}(u)\, v^\alpha v^\beta = ||v||^2.$$

Thus we have

$$a_\sigma(x, y) = 1 + \sigma \left[1 - \frac{1}{\eta^2(x)}\right] ||y||^2 = 1 + \sigma \left[1 - \frac{1}{\eta^2(x)}\right] ||v||^2.$$

Therefore the induced $d-$tensor field $C^\alpha_{\beta\gamma}$ is obtained from (2.6), (3.10)(b) and (3.11)(c), which is given by

$$(4.7) \qquad C^\alpha_{\beta\gamma} = \frac{1}{a_1}\left(1 - \frac{1}{\eta^2(x)}\right) \gamma_{\beta\gamma} v^\alpha.$$

The intrinsic $d-$tensor field $\underline{C}^{\alpha}_{\beta\gamma}$ of $G\overline{L}^m$ is defined from induced $g_{\alpha\beta}$ by

$$\underline{C}^{\alpha}_{\beta\gamma} = g^{\alpha\delta}\underline{C}_{\beta\delta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(\frac{\partial g_{\beta\delta}}{\partial v^{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial v^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial v^{\delta}}\right).$$

Thus from (4.4), we have

$$(4.8) \qquad \underline{C}^{\alpha}_{\beta\gamma} = \frac{1}{a_1}\left(1 - \frac{1}{\eta^2(x)}\right)\gamma_{\beta\gamma}v^{\alpha}.$$

Hence from (4.7) and (4.8) we have the following :

Theorem (4.2). The induced $d-$tensor field $C^{\alpha}_{\beta\gamma}$ of $G\overline{L}^m$ is identical with the intrinsic $d-$tensor field $\underline{C}^{\alpha}_{\beta\gamma}$ of $G\overline{L}^m$.

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