On Cyclic Ricci-recurrent Spaces

A. A. Shaikh, S. K. Hui and A. Patra

Department of Mathematics,
The University of Burdwan, Burdwan, 713104, West Bengal, India

e-mail: aask2003@yahoo.co.in, skhui@math.buruniv.ac.in, akshoyp@gmail.com

(Received: November 18, 2019, Accepted: December 27, 2019)
(Dedicated to the memory of the late Professor Witold Roter)

Abstract

The object of the present paper is to study cyclic Ricci-recurrent spaces. Some basic geometric properties of such a space are obtained. Among others we study conformally symmetric cyclic Ricci-recurrent spaces. Also we study decomposibility and conformal deformation of cyclic Ricci-recurrent spaces. Finally, the existence of such space is ensured by an interesting example.

Keywords: Ricci symmetric, Ricci-recurrent, cyclic Ricci parallel, cyclic Ricci-recurrent, conformally symmetric, conformally flat space, scalar curvature, Codazzi tensor.

2010 AMS Subject Classification : 53B30, 53C15.

1. Introduction

Let $M^n$ be a Riemannian space of dimension $n$ with Riemannian metric $g$. Then $M^n$ is said to be Ricci symmetric or Ricci parallel if its Ricci tensor $S_{ij}$ of type $(0, 2)$ satisfies the condition $S_{ij,k} = 0$, where ',' denotes the covariant differentiation with respect to the metric tensor $g$. The class of Ricci parallel spaces is very natural generalization of the class of spaces of constant scalar curvature. Again generalizing the notion of Ricci parallel space, Patterson [5] introduced the notion of Ricci-recurrent space and later studied by Roter [6] and various authors. A Riemannian space is said to be Ricci-recurrent if $S_{ij}$ satisfies the condition $S_{ij,k} = A_k S_{ij}$, where $A_k$ is a nowhere vanishing 1-form. Again by the decomposition of the covariant derivative $S_{ij,k}$ of $S_{ij}$, Gray [4] introduced two important classes $A$, $B$, which lie between the class of Ricci-parallel spaces

\footnote{Corresponding author}
and the spaces of constant scalar curvature, namely (i) the class $\mathcal{A}$ is the class of spaces whose Ricci tensor is cyclic parallel and (ii) the class $\mathcal{B}$ is the class of spaces whose Ricci tensor is of Codazzi type. The spaces of class $\mathcal{A}$ are said to be cyclic Ricci parallel spaces. Generalizing the notion of cyclic Ricci parallel space, in the present paper we introduce the notion of cyclic Ricci-recurrent space. A Riemannian space $M^n, (n > 2)$ is said to be cyclic Ricci-recurrent if $S_{ij}$ satisfies the condition

$$S_{ij,k} + S_{jk,i} + S_{ki,j} = A_k S_{ij} \text{ (or } = A_i S_{jk} \text{ or } = A_j S_{ki}), \quad (1.1)$$

where $A_k$ is a nowhere vanishing 1-form associated to the vector field $\rho$ such that $A_k = \rho^m g_{km}$.

The present paper is organized as follows. Section 2 deals with preliminaries. Section 3 is concerned with some basic properties of cyclic Ricci-recurrent spaces. It is proved that if the Ricci tensor of a cyclic Ricci-recurrent space is a Codazzi one then the scalar curvature of the space vanishes. Also it is shown that a cyclic Ricci-recurrent space is Ricci-recurrent if and only if its Ricci tensor is a Codazzi one.

In section 4, we investigate conformally symmetric cyclic Ricci-recurrent spaces and proved that a conformally symmetric cyclic Ricci-recurrent space is Ricci-recurrent if and only if its scalar curvature vanishes. Section 5 deals with decomposibility of cyclic Ricci-recurrent space and it is shown that a decomposable Riemannian space is cyclic Ricci-recurrent if and only if one of the decomposition spaces is cyclic Ricci-recurrent and the other is Ricci flat. Section 6 is devoted to the study of conformal deformation of cyclic Ricci-recurrent space. Finally, the last section provides the existence of a cyclic Ricci-recurrent space with vanishing scalar curvature.

2. Preliminaries

In the sequel each Latin index runs over $1, 2, \cdots, n$ and each Greek index runs over $2, 3, \cdots, n - 1$ and we shall need the following results:

Transvecting (1.1) with $g^{ij}$ and using the well known relation $S_{j,r}^r = \frac{1}{2}K_j$, we get

$$2K_j = KA_j, \quad A_r S_j^r = 2K_j, \quad (2.1)$$

where $K$ is the scalar curvature of the space.
Moreover, using the condition \( R^{r}_{ij,k,r} = S_{ij,k} - S_{ik,j} \) (which is a consequence of the second Bianchi identity) and (1.1) we obtain
\[
S_{ij,l} + R^{r}_{ijl,r} + R^{r}_{jil,r} = 4S_{ij,l} - A_l S_{ij} \tag{2.2}
\]
and
\[
S_{il,j} + S_{lj,i} = A_l S_{ij} - S_{ij,l}. \tag{2.3}
\]
From (1.1) we have
\[
A_k S_{ij} = A_i S_{jk} \tag{2.4}
\]
hence
\[
S_{ij} = A_i \Theta_j \tag{2.5}
\]
where \( \Theta_j = \tau^k S_{kj} \) and \( \tau \) is so chosen that \( \tau^r A_r = 1 \).

Since the Ricci tensor is symmetric, we get
\[
S_{ij} = p A_i A_j, \tag{2.6}
\]
where \( p = \tau^r \Theta_r \). Hence we have
\[
\text{rank } S_{ij} \leq 1. \tag{2.7}
\]

We now suppose that the Ricci tensor does not vanish. Then putting
\[
B_j = \sqrt{\epsilon p} A_j, \tag{2.8}
\]
where \( \epsilon = 1 \) or \(-1\), we obtain from (2.6) that
\[
S_{ij} = \epsilon B_i B_j. \tag{2.9}
\]

**Lemma 2.1.** Let us assume that \( B_i \) and \( T_{jk} \) are numbers satisfying
\[
B_i T_{jk} + B_j T_{ik} = 0. \tag{2.10}
\]
If not all \( B_i \) are zero then \( T_{jk} = 0 \) for \( j, k = 1, 2, \ldots, n \).

**Proof.** Let \( B_\alpha \neq 0 \). Putting \( i = j = \alpha \) in (2.10) we get \( B_\alpha T_{\alpha k} = 0 \), whence \( T_{\alpha k} = 0 \) for all \( k \).

Now let \( i = \alpha \) then we have \( B_\alpha T_{jk} = 0 \) and hence \( T_{jk} = 0 \) for all \( j \) and \( k \).

### 3. Some basic properties of cyclic Ricci-recurrent Spaces

This section deals with various basic properties of cyclic Ricci-recurrent spaces.

**Theorem 3.1.** If the Ricci tensor of a cyclic Ricci-recurrent space \( M \) is a Codazzi one then the scalar curvature \( K \) of the space vanishes.
Proof. Obviously if at a point of $M$ the Ricci tensor vanishes then at that point $K = 0$. Therefore suppose that $S_{ij} \neq 0$.

Since the Ricci tensor is of Codazzi type [3] then we have

$$S_{ij,k} = S_{ik,j} = S_{kj,i},$$  \hspace{1cm} (3.1)

which yields

$$K, j = 0 \text{ for all } j.$$ \hspace{1cm} (3.2)

Using (3.1) in (1.1) we obtain

$$S_{ij,k} = \frac{1}{3} A_k S_{ij}.$$ \hspace{1cm} (3.3)

Differentiating (2.9) covariantly and using (3.3) we get

$$B_i B_j, k + B_j B_i, k = \frac{1}{3} A_k S_{ij},$$ \hspace{1cm} (3.4)

which can be written as

$$B_i (B_{j,k} - \frac{1}{6} A_k B_j) + B_j (B_{i,k} - \frac{1}{6} A_k B_i) = 0.$$ \hspace{1cm} (3.5)

In view of Lemma 2.1 it follows from (3.5) that

$$B_{j,k} = \frac{1}{6} A_k B_j.$$ \hspace{1cm} (3.6)

Again from (2.9) we get

$$K = \epsilon B^r B_r.$$ \hspace{1cm} (3.7)

Differentiating (3.7) covariantly and using (3.2) we obtain

$$2 \epsilon B^r B_{r,k} = 0.$$ \hspace{1cm} (3.8)

Using (3.6) in (3.8) we get

$$\frac{1}{3} \epsilon A_k B^r B_r = 0.$$ \hspace{1cm} (3.9)

Since $A_k \neq 0$ therefore from (3.7) and (3.9) we have $K = 0$ and hence the proof is complete.

We now assume that a cyclic Ricci-recurrent space is Ricci-recurrent and the Ricci tensor does not vanish at every point of a subset $U$ of $M$. Then in view of (2.9) and

$$S_{ij,k} = \Phi_k S_{ij},$$ \hspace{1cm} (3.10)

we obtain

$$B_i B_{j,k} + B_j B_{i,k} = \Phi_k B_i B_j,$$ \hspace{1cm} (3.11)
which yields by Lemma 2.1

\[ B_i B_{j,k} = \frac{1}{2} \Phi_k B_j. \]  

(3.12)

In view of (2.9) and (3.10) it follows from (1.1) that

\[ B_i B_j \Phi_k + B_k B_i \Phi_j + B_k B_i \Phi_j = q B_i B_j B_k, \]  

(3.13)

where \( q = \frac{1}{\sqrt{ep}} \). But (3.13) can be written as

\[ B_i B_j a_k + B_j B_k a_i + B_i B_k a_j = 0, \]  

(3.14)

where \( a_j = \Phi_j - \frac{1}{3} q B_j \).

Suppose now \( a_\alpha \neq 0 \) then (3.14) implies that \( 3 B_\alpha B_\alpha a_\alpha = 0 \) and hence

\[ B_\alpha = 0. \]  

Moreover putting \( k = \alpha \) in (3.14) and using the last result we obtain \( a_\alpha B_i B_j = 0 \), which yields \( B_j = 0 \), a contradiction. Thus \( a_\alpha \) must be equal to zero and hence

\[ \Phi_j = \frac{1}{3} q B_j. \]  

(3.15)

Using (3.15) in (3.12) we get

\[ B_{j,k} = \frac{1}{6} q B_j B_k \]  

(3.16)

From (2.9) and (3.16) we have

\[ S_{i,j,k} = S_{i,k,j}, \]  

(3.17)

which implies that the Ricci tensor of a cyclic Ricci-recurrent space is at \( U \) a Codazzi one.

Suppose now that the Ricci tensor vanishes at some point \( x \) of \( M \). Then from (3.10) we obtain

\[ S_{i,j,k} = 0 = S_{i,k,j}, \]  

(3.18)

that is, the Ricci tensor of a cyclic Ricci-recurrent space is at \( x \) a Codazzi one. Thus if a cyclic Ricci-recurrent space is Ricci-recurrent then the Ricci tensor of the space is Codazzi one. Also from (3.3) it follows that if the Ricci tensor of a cyclic Ricci-recurrent space is Codazzi one then the space is Ricci-recurrent. Hence we can state the following:

**Theorem 3.2.** A cyclic Ricci-recurrent space is Ricci-recurrent if and only if its Ricci tensor is a Codazzi one.
Moreover we now consider that $A_j$ is locally a gradient, that is, $A_{i,j} = A_{j,i}$. Then there exists a function, say $A$, such that $A_j = A_i$.

Define now $\psi$ as follows:

$$\psi_j = e^{-\frac{1}{6}A}B_j.$$

(3.19)

Then in view of (3.6) we have

$$\psi_{j,k} = 0.$$  

(3.20)

Also from (3.19) we get

$$\psi^r \psi_r = e^{-\frac{1}{6}A}B^rB_r = 0,$$

(3.21)

which is an immediate consequence of (2.9) and hence $K = 0$. Thus we can state the following:

**Theorem 3.3.** If the covector $A$ of a cyclic Ricci-recurrent space is locally a gradient and its Ricci tensor does not vanish and it is a Codazzi one then the manifold is Ricci-recurrent and it admits locally a null parallel vector field.

4. Conformally Symmetric cyclic Ricci-recurrent Spaces

This section deals with conformally symmetric and conformally flat cyclic Ricci-recurrent spaces.

**Theorem 4.1.** Let $(M, g)$ be conformally symmetric cyclic Ricci-recurrent space. Then $M$ is Ricci-recurrent if and only if the scalar curvature of $M$ vanishes.

**Proof.** The Weyl conformal curvature tensor $C_{hijk}$ of type $(0, 4)$ is given by

$$C_{hijk} = R_{hijk} - \frac{1}{n-2} [S_{ij}g_{hk} - S_{hj}g_{ik} + S_{hk}g_{ij} - S_{ik}g_{hj}]$$

(4.1)

and hence

$$C_{hijk,l} = R_{hijk,l} - \frac{1}{n-2} [S_{ij}g_{hk}g_{l} - S_{hj}g_{ik} + S_{hk}g_{ij}g_{l} - S_{ik}g_{hj}g_{l}]$$

(4.2)

$$+ \frac{1}{(n-1)(n-2)} [g_{ij}g_{hk}g_{l} - g_{hj}g_{ik}g_{l}],$$

Since the space is conformally symmetric, we have [2]

$$C_{hijk,l} = 0$$

(4.3)

and hence

$$C_{hijk,l} + C_{kij,l} + C_{lij,k} = 0.$$  

(4.4)
In view of (4.2) and (1.1), (4.4) yields
\[ R_{hijk,l} + R_{kijl,h} + R_{lijh,k} - \frac{1}{n-2} \left[ A_l g_{ij} S_{hk} - g_{ik} S_{hj,l} + g_{hk} S_{ij,l} - g_{ij} S_{hk,l} \right] + \frac{1}{(n-1)(n-2)} \] (4.5)

Transvecting (4.5) with \( g^{hk} \) and making use of Lemma 2.1, we get
\[ 4 S_{ij,l} - A_l S_{ij} - \frac{1}{n-2} [2 K_{l,i} g_{ij} + n S_{ij,l} - \frac{1}{2} K_{j} g_{il} - \frac{1}{2} K_{i} g_{jl} - S_{ij,l} - A_l S_{ij}] \]
\[ + \frac{1}{(n-1)(n-2)} [(n+1) K_{l,i} g_{ij} - K_{i} g_{ij} - K_{j} g_{il}] = 0, (4.6) \]

whence, by a quite elementary computation, we find
\[ (n-3)[3(n-1)] S_{ij,l} - (n-1) A_l S_{ij} - K_{l,i} g_{ij} + \frac{1}{2} K_{j} g_{il} + \frac{1}{2} K_{i} g_{jl} = 0. (4.7) \]

Consequently we have finally
\[ 3(n-1) S_{ij,l} - (n-1) A_l S_{ij} - K_{l,i} g_{ij} + \frac{1}{2} K_{j} g_{il} + \frac{1}{2} K_{i} g_{jl} = 0. (4.8) \]

If \( K_{j} = 0 \) or by (2.1), \( K = 0 \) then from (4.8) we get
\[ S_{ij,l} = \frac{1}{3} A_l S_{ij}, \] (4.9)

that is the space is Ricci-recurrent. Conversely, if the space is Ricci-recurrent then (4.9) holds and using (4.9) in (4.8) we obtain \( K_{j} = 0 \) and hence from (2.1) we have \( K = 0 \). This completes the proof.

**Corollary 4.1.** Let \( M \) be conformally flat cyclic Ricci-recurrent space. Then \( M \) is Ricci-recurrent if and only if the scalar curvature of \( M \) vanishes.

**Proof.** Evidently every conformally flat space is conformally symmetric. Hence the relation (4.8) holds for conformally flat cyclic Ricci-recurrent spaces. Thus every conformally flat cyclic Ricci-recurrent space with vanishing scalar curvature is Ricci-recurrent. Conversely, if a conformally flat and cyclic Ricci-recurrent space is Ricci-recurrent then, which follows from (4.8), the space must have a vanishing scalar curvature.

**Theorem 4.2.** (i) In a conformally symmetric cyclic Ricci-recurrent space with vanishing scalar curvature, coordinates can be locally choosen so that the metric
takes the form
\[
ds^2 = \Omega(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1dx^n, \tag{4.10}
\]
\[
\Omega = (Gk_{\lambda\mu} + a_{\lambda\mu})x^\lambda x^\mu, \tag{4.11}
\]
where \([k_{\lambda\mu}]\) is a symmetric and non-singular matrix of constants, \(a_{\lambda\mu}\) is a symmetric matrix of constants satisfying
\[
k^\lambda \omega a_{\lambda\omega} = 0 \text{ with } [k^\lambda] = [k_{\lambda\omega}]^{-1}
\]
and \(G\) is a non-zero and non-constant function of \(x^1\) only.

(ii) Let \(\mathbb{R}^n\) be endowed with the metric satisfying (4.10) and (4.11), where \([k_{\lambda\mu}]\) and \([a_{\lambda\mu}]\) are as above and \(G\) is a function of \(x^1\) only such that \(0 \neq G \neq \text{constant}\). Then \(\mathbb{R}^n\) is conformally symmetric and cyclic Ricci-recurrent. Moreover its scalar curvature vanishes.

**Proof.** Assume that the scalar curvature of a conformally symmetric cyclic Ricci-recurrent space \(M\) vanishes. Then by Theorem 4.1, the space \(M\) is Ricci-recurrent with recurrence vector field \(\tau_j = \frac{1}{3}A_j\).

Here and in the sequel we assume that the Ricci tensor does not vanish.

Adati and Miyazawa [1] proved that in a conformally symmetric Ricci-recurrent space \(\tau_j\) is locally a gradient. Hence there exists a function, say \(\tau\) such that \(\tau_j = \tau_j\).

Define now \(\psi\) as follows:
\[
\psi_j = e^{-\frac{1}{2}\tau}B_j, \tag{4.12}
\]
where \(B_j\) satisfies (3.12). But by Theorem 3.3 it follows that \(\psi_j\) is parallel and null. Therefore \(M\) admits locally a null parallel vector field.

From (2.6) and (2.9) we have
\[
\epsilon B_iB_j = pA_iA_j,
\]
whence \(A_j = \sigma B_j\). Hence by (4.12) and \(\tau_j = \frac{1}{3}A_j\) we get \(\tau_j = \alpha\psi_j\). Therefore the recurrence vector field \(\tau_j\) is codirectional with a parallel null vector field \(\psi_j\).

From [6] it follows that the curvature tensor of a conformally symmetric Ricci-recurrent space has locally the form
\[
R_{jhkm} = \tau_h\tau_kS_{mj} - \tau_h\tau_jS_{mk} + \tau_m\tau_jS_{hk} - \tau_m\tau_kS_{hj}, \tag{4.13}
\]
where
\[
S_{ij} = S_{ji} = a^r a^s R_{r_1j s}, \quad \text{and} \quad a^r \tau_r = 1.
\]
Again Walker ([7],[8]) proved that if a pseudo-Riemannian space with the curvature tensor of the form (4.13) admits a null parallel vector field $\psi^i$ satisfying $\tau_j = \alpha \psi_j$ then one can choose coordinates so that the metric can be written as

$$ds^2 = \theta(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1dx^n,$$  \hspace{1cm} (4.14)

where $k_{\lambda\mu}$ are constants, det$[k_{\lambda\mu}] \neq 0$ and $\theta$ is independent of $x^n$.

In this coordinate system the null parallel vector field is of the form $\psi^i = \delta^i_n$, whence in view of $\tau_j = \alpha \psi_j$, we have

$$\tau_j = \alpha g_{ij} \psi^i = \alpha g_{jn} = \alpha \delta^1_j.$$  

The recurrence vector $\tau_j$ is therefore a gradient of some function $\tau(x^1)$ and so $\alpha$ is a function of $x^1$ only.

In the metric (4.14) the only components of $R$ and $S$ not identically zero are these related to

$$R_{1\lambda\mu1} = \frac{1}{2} \theta_{,\lambda\mu}, \quad S_{11} = \frac{1}{2} k^{\beta\omega} \theta_{,\beta\omega},$$  \hspace{1cm} (4.15)

where the dot denotes partial differentiation with respect to coordinates.

Moreover one can easily show that

$$C_{1\lambda\mu1} = \frac{1}{2} \theta_{,\lambda\mu} - \frac{1}{2(n-2)} k_{\lambda\mu} k^{\beta\omega} \theta_{,\beta\omega}, \quad R_{1\lambda\mu1,j} = \frac{1}{2} \theta_{,\lambda\mu j}$$  \hspace{1cm} (4.16)

and

$$C_{1\lambda\mu1,j} = \frac{1}{2} \theta_{,\lambda\mu j} - \frac{1}{2(n-2)} k_{\lambda\mu} k^{\beta\omega} \theta_{,\beta\omega j}, \quad S_{11,j} = \frac{1}{2} k^{\beta\omega} \theta_{,\beta\omega j}.$$  \hspace{1cm} (4.17)

All other components of $C$ and the covariant derivative of $S, R$ and $C$ are identically zero.

Since the space is, by assumption, conformally symmetric and Ricci-recurrent we obtain

$$\theta_{,\lambda\mu j} = \frac{1}{n-2} k_{\lambda\mu} (k^{\beta\omega} \theta_{,\beta\omega})_{,j},$$  \hspace{1cm} (4.18)

$$(k^{\beta\omega} \theta_{,\beta\omega})_{,j} = \alpha \delta^1_j k^{\beta\omega} \theta_{,\beta\omega}.$$  \hspace{1cm} (4.19)

From (4.18) and (4.19) we have

$$\theta_{,\lambda\mu} = 2Gk_{\lambda\mu} + 2a_{\lambda\mu},$$  \hspace{1cm} (4.20)

where $a_{\lambda\mu}$ are constants such that $k^{\beta\omega} a_{\beta\omega} = 0$ and $G$ is a function of $x^1$ only. Hence

$$\theta = Gk_{\lambda\mu} x^\lambda x^\mu + a_{\lambda\mu} x^\lambda x^\mu + k_{\lambda\mu} + \chi,$$  \hspace{1cm} (4.21)
Consider now a transformation [7] of the form

\[ x'_{\lambda} = x_{\lambda} - k_{\lambda \mu} \sigma_{\mu}, \quad x'_n = x_n + \rho_{\lambda} x_{\lambda} + \eta \]  

(4.22)

from \( x^2, x^3, \ldots, x^n, x'^2, x'^3, \ldots, x'^n \), where \( \rho_{\lambda}, \sigma_{\lambda} \) and \( \eta \) are functions of \( x^1 \) satisfying

\[ \rho_{\lambda} = \frac{1}{2} \int k_{\lambda} dx^1, \quad \sigma_{\lambda} = \int \rho_{\lambda} dx^1, \quad \eta = \frac{1}{2} \int (\chi + k_{\beta \omega} \theta_{\beta \omega}) dx^1. \]  

(4.23)

Transforming (4.14) and (4.21) and omitting the primes, we obtain (4.10) and (4.11) for the metric of a conformally symmetric cyclic Ricci-recurrent space.

(ii) From (4.20) it follows that

\[ G = \frac{1}{2(n-2)} k_{\beta \omega} \theta_{\beta \omega}. \]

But (4.17) implies

\[ S_{11,1} = \frac{1}{G} G_{,1} S_{11}. \]

The last condition shows that the space is Ricci-recurrent and the Ricci tensor of this space is a Codazzi one. Hence \( \mathbb{R}^n \) is cyclic Ricci-recurrent. Moreover from (4.17) and (4.20) it follows that \( \mathbb{R}^n \) is also conformally symmetric and because of \( g^{11} = 0 \) and \( K = g_{ij} S_{ij} = g^{11} S_{11} = 0 \), its scalar curvature vanishes. This completes the proof.

5. Decomposable Cyclic Ricci-recurrent Spaces

A Riemannian space \( M^n \) is decomposable [9] if it can be expressed as a product \( M^p_1 \times M^{n-p}_2 \) for some \( p(2 \leq p \leq n-2) \), i.e., if coordinates can be found so that its metric takes the form

\[ ds^2 = \sum_{a,b=1}^{p} g_{ab} dx^a dx^b + \sum_{\alpha,\beta=p+1}^{n} g_{\alpha\beta} dx^\alpha dx^\beta, \]  

(5.1)

where the \( g_{ab} \) are functions of \( x^1, x^2, \ldots, x^p \) only and the \( g_{\alpha\beta} \) are functions of \( x^{p+1}, x^{p+2}, \ldots, x^n \) only. Latin letters \( a, b, c, d, \cdots \) range over the indices \( 1, 2, \cdots, p \) and Greek letters \( \alpha, \beta, \gamma, \delta, \cdots \) range over the indices \( p+1, p+2, \cdots, n \). The two parts of (5.1) are the metrics of \( M^p_1 \) and \( M^{n-p}_2 \), called the decomposition spaces of \( M^n \). From the above form of the metric it is easily seen that the Christoffel symbols and the components of the curvature tensor and its covariant derivatives in \( M^n \) are zero unless all suffixes belong to the same range \( 1, 2, \cdots, p \) or \( p+1, p+2, \cdots, n \). When all the suffixes belong to the same range, say
1, 2, \cdots, p$, then the symbols and the tensor components are the same for $M^n_p$ as for $M^n$ and covariant differentiation in $M^n_p$ is the same as in $M^n$ with respect to $x^1, x^2, \cdots, x^p$. When one of the decomposition spaces, say, $M^{n-p}_2$ is flat then $M^n$ described as a flat extension of $M^n_p$.

Let us consider a decomposable Riemannian space $M^n = M^n_p \times M^{n-p}_2 (2 \leq p \leq n-2)$ which is cyclic Ricci-recurrent. Then we have from (1.1) that

\[
S_{a,b,c} + S_{b,c,a} + S_{c,a,b} = A_c S_{a,b},
\]

(5.2)

\[
S_{a,\beta,\gamma} + S_{\beta,\gamma,a} + S_{\gamma,a,\beta} = A_\gamma S_{a,\beta}.
\]

(5.3)

Taking $c = \gamma$ in (5.2) we get

\[
A_\gamma S_{a,b} = 0,
\]

(5.4)

which implies that $S_{a,b} = 0$, since $A_\gamma \neq 0$ and hence the decomposition $M^n_p$ is Ricci flat and the decomposition $M^{n-p}_2$ is cyclic Ricci-recurrent. Again taking $\gamma = c$ in (5.3) we obtain $M^{n-p}_2$ is Ricci flat and $M^n_p$ is cyclic Ricci-recurrent. Conversely, if $M^n_p$ is Ricci flat and $M^{n-p}_2$ is cyclic Ricci-recurrent then $M^n = M^n_p \times M^{n-p}_2$ is cyclic Ricci-recurrent. This leads to the following:

**Theorem 5.1.** Let $M^n = M^n_p \times M^{n-p}_2$ be a decomposable Riemannian space. Then $M$ is cyclic Ricci-recurrent if and only if one of the decomposition spaces is cyclic Ricci-recurrent and the other is Ricci flat.

### 6. Conformal Mapping of cyclic Ricci-recurrent Spaces

Let $M$ be an $n$-dimensional smooth space with metric tensors $g$ and $\bar{g}$ relative to a neighbourhood $U$ with local coordinates $x^i$, we have

\[
g_{ij} = e^{2\sigma} g_{ij},
\]

(6.1)

where $\sigma$ is a smooth function of the coordinates $x^i$. Clearly the angle between any two directions at point of $U$ is independent of the choice of metric $g$ or $\bar{g}$. We say that these two spaces $(M, g)$ and $(M, \bar{g})$ are conformally related. From (6.1) it follows that

\[
\bar{g}^{ij} = e^{-2\sigma} g^{ij}.
\]

(6.2)

A straightforward calculation shows that the Christoffel symbols are related by

\[
\Gamma^l_{ij} = e^{2\sigma} ([ij, k] + g_{ik} \sigma_j + g_{jk} \sigma_i - g_{ij} \sigma_k),
\]

(6.3)

\[
\bar{\Gamma}^l_{ij} = \Gamma^l_{ij} + \delta^l_j \sigma, i + \delta^l_i \sigma, j - g_{ij} g^{lm} \sigma, m,
\]

(6.4)

where $\sigma, i = \frac{\partial \sigma}{\partial x^i}$ and $[ij, k] = g_{hk} \Gamma^h_{ij}$.
In Eisenhart’s notation, the covariant form of the curvature tensor has components

\[ R_{hijk} = \frac{\partial}{\partial x^j}[ik,h] - \frac{\partial}{\partial x^k}[ij,h] + \Gamma^l_{ij}[hk,l] - \Gamma^l_{ik}[hj,l]. \]  

(6.5)

If we substitute for the analogous expression derived from \( \bar{g} \) we find

\[ e^{-2\sigma} \bar{R}_{hijk} = R_{hijk} + g_{hk} \sigma_{ij} + g_{ij} \sigma_{hk} - g_{hj} \sigma_{ik} \]

(6.6)

where

\[ \sigma_{ij} = \sigma_{ij}^l - \sigma_{i}^l \sigma_{j}^l \]  

(6.7)

and \( \Delta_1 \sigma \) is the first Beltrami operator defined by

\[ \Delta_1 \sigma = g^{ij} \sigma_{,ij}. \]  

(6.8)

Contracting (6.6) over the indices \( h \) and \( k \) and using (6.2) we obtain

\[ \bar{S}_{ij} = S_{ij} + (n - 2) \sigma_{ij} + [\Delta_2 \sigma + (n - 2) \Delta_1 \sigma] g_{ij}, \]  

(6.9)

where \( \Delta_2 \sigma \) is the second Beltrami operator defined by

\[ \Delta_2 \sigma = g^{ij} \sigma_{,ij}. \]  

(6.10)

Again taking contraction of (6.9) we obtain

\[ \bar{K} = e^{-2\sigma} [K + 2(n - 1) \Delta_2 \sigma + (n - 1)(n - 2) \Delta_1 \sigma]. \]  

(6.11)

We now suppose that both \((M, g)\) and \((M, \bar{g})\) are cyclic Ricci-recurrent spaces. Then we have the relation (1.1) and

\[ \bar{S}_{ij,k} + \bar{S}_{jk,i} + \bar{S}_{ki,j} = \bar{A}_k \bar{S}_{ij} \] \( \text{or} \) \( = \bar{A}_i \bar{S}_{jk} \) \( \text{or} \) \( = \bar{A}_j \bar{S}_{ki} \),  

(6.12)

where \( \bar{A}_k \) is a nowhere vanishing 1-form such that \( \bar{A}_k = \rho^m g_{km} \).

From (6.9) we have

\[ \bar{S}_{ij} = S_{ij} + 2(n - 1) \sigma_{ij} + n(n - 1) \sigma_{,i} \sigma_{,j} \]  

(6.13)

and hence

\[ \bar{S}_{ij,k} = S_{ij,k} + 2(n - 1) \sigma_{ij,k} + n(n - 1) \{ \sigma_{,ik} \sigma_{,j} + \sigma_{,i} \sigma_{,jk} \}. \]  

(6.14)

By virtue of (6.14) we obtain from (6.12) that

\[ (\bar{A}_k - A_k) S_{ij} = (n - 1) \{ 2(\sigma_{ij,k} + \sigma_{jk,i} + \sigma_{ki,j}) \} + n(\sigma_{,ik} \sigma_{,j} + \sigma_{,i} \sigma_{,jk} + \sigma_{,j} \sigma_{,ik} + \sigma_{,k} \sigma_{,ij}) - \{ 2\sigma_{,ij} + (n - 2) \sigma_{,i} \sigma_{,j} \} \bar{A}_k \].  

(6.15)
We may assume that $\bar{A}_k = A_k$, then (6.15) yields

$$\{2\sigma_{ij} + (n-2)\sigma_{i}\sigma_{j}\}\bar{A}_k = 2(\sigma_{ij,k} + \sigma_{jk,i} + \sigma_{ki,j})$$

$$+ 2n(\sigma_{ij}\sigma_{,k} + \sigma_{jk}\sigma_{,i} + \sigma_{ki}\sigma_{,j} + 3\sigma_{i}\sigma_{j}\sigma_{,k}).$$

Again if the 1-form $\bar{A}_k$ is of the form (6.16) then from (6.15) we get $\bar{A}_k = A_k$, that is, the 1-form $A_k$ of the space is invariant. This leads to the following:

**Theorem 6.1.** If a cyclic Ricci-recurrent space is transformed into another cyclic Ricci-recurrent space then the associated 1-form of the space is invariant if and only if the 1-form of the space satisfies the relation (6.16).

Contracting (6.12) over $i$ and $k$ we obtain

$$2\bar{K}_j = \bar{K}\bar{A}_j.$$

From (6.11) we have

$$\bar{K} = e^{-2\sigma} K + 2(n-1)\bar{g}^{ik}\sigma_{ik} + n(n-1)\bar{g}^{ik}\sigma_{,i}\sigma_{,k}$$

and hence

$$\bar{K}_j = -2e^{-2\sigma}\sigma_{j}K + e^{-2\sigma} K_{,j} + 2(n-1)\bar{g}^{ik}\sigma_{ik,j}$$

$$+ n(n-1)\bar{g}^{ik}[\sigma_{,ij}\sigma_{,k} + \sigma_{i}\sigma_{,kj}].$$

In view of (6.19), (6.17) yields

$$-4e^{-2\sigma}\sigma_{j}K + 2e^{-2\sigma} K_{,j} + 4(n-1)\bar{g}^{ik}\sigma_{ik,j}$$

$$+ 2n(n-1)\bar{g}^{ik}[\sigma_{,ij}\sigma_{,k} + \sigma_{i}\sigma_{,kj}] = \bar{K}\bar{A}_j.$$

This leads to the following:

**Theorem 6.2.** If a cyclic Ricci-recurrent space is transformed into another cyclic Ricci-recurrent space then the associated 1-form of the space satisfies the relation (6.20).

7. **Example of Cyclic Ricci-recurrent Space**

This section deals with an interesting example of cyclic Ricci-recurrent space.

**Example 7.1.** Let $\mathbb{R}^n (n > 3)$ be endowed with the following metric

$$g_{ij}dx^idx^j = \phi(dx^i)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1dx^n,$$

$$\phi = (Ak_{\lambda\mu} + Dc_{\lambda\mu})x^\lambda x^\mu,$$
where \([k_{\lambda\mu}]\) is a symmetric and non-singular matrix consisting of constants, \([c_{\lambda\mu}]\) is a symmetric matrix of constants satisfying \(\text{rank} c_{\lambda\mu} > 1\) and \(k_{\lambda\mu} c_{\lambda\mu} = 0\) with \([k_{\lambda\mu}] = [k_{\lambda\mu}]^{-1}\) and \(A, D\) are functions of \(x^1\) only such that \(0 \neq A \neq \text{constant}, 0 \neq D \neq \text{constant}\). Then \(\mathbb{R}^n\) with above metric is conformally recurrent and Ricci-recurrent with vanishing scalar curvature. Moreover it is also cyclic Ricci-recurrent and its Ricci tensor is a Codazzi one.

**Proof.** In the above metric the only component of the Ricci tensor, Weyl conformal curvature tensor and their covariant derivatives not identically zero are those related to

\[
S_{11} = (n - 2)A, \quad C_{1\lambda\mu 1} = Dc_{\lambda\mu},
\]

\[
S_{11,j} = (n - 2)A_j, \quad C_{1\lambda\mu 1,j} = D_jc_{\lambda\mu}.
\]

Moreover as one can easily verify, in the metric (7.1) we have \(g^{11} = 0\). Hence the scalar curvature \(K = g^{ij}S_{ij} = g^{11}S_{11} = 0\). The assertion is now a consequence of (7.3), (7.4) and (1.1). This completes the proof.

The above example shows that there exists a subclass of cyclic Ricci-recurrent metrics with vanishing scalar curvature. Thus we can state the following:

**Theorem 7.1.** There exists a cyclic Ricci-recurrent space with vanishing scalar curvature.

**Acknowledgement.** The work was initiated when the first author visited Wroclaw University of Science and Technology in 2011 and made discussion about the work with Professor Witold Roter.

**References**


