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Some Curvature Tensors of a Semi Symmetric Metric $\phi-$Connection in an LSP-Sasakian Manifold

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Abstract

The purpose of this paper is to investigate the conditions for the pseudo-projective curvature tensor and quasi-conformal curvature tensor of a semi-symmetric metric ϕ-connection to be the pseudo-projective curvature tensor and quasi-conformal curvature tensor of a Levi-Civita connection on LSP-Sasakian manifold. Also we shall discuss the behavior of conharmonic curvature tensor and Tachibana concircular curvature tensor with respect to semi-symmetric metric ϕ-connection on LSP-Sasakian manifold.

Key Words : Semi-symmetric metric ϕ-connection, LSP-Sasakian manifold, Quasi-conformal curvature tensor, Pseudo-projective curvature tensor, Conharmonic curvature tensor, Tachibana concircular curvature tensor.
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1. Introduction

The definition of semi-symmetric metric ϕ-connection has been introduced by K.Yano [5]. Also he discussed some properties on semi-symmetric metric ϕ-connection. Some results on semi-symmetric metric connection on Sasakian manifold developed by K.Yano and T.Imai [8]. Later V.K.Jaiswal et. al. studied some properties on curvature tensor and projective curvature tensor on semi-symmetric ϕ-connection in Lorentzian Special Para-Sasakian manifold [7]. In this paper we shall show the necessary and sufficient condition for pseudo projective curvature tensor and quasi conformal curvature tensor with semi-symmetric metric ϕ-connection $\bar{D}$ to be pseudo projective curvature tensor and quasi conformal curvature tensor with Levi-Civita connection on LSP-Sasakian manifold respectively. Also our aim is to find conditions for which conharmonic curvature tensor and Tachibana curvature tensor with semi-symmetric metric ϕ-connection $\bar{D}$ equals to conharmonic curvature tensor and Tachibana curvature tensor with Levi-Civita connection on LSP-Sasakian manifold.

2. Preliminaries

An n-dimensional differentiable manifold M is said to a Lorentzian Para-Sasakian (LP-Sasakian) manifold if it admits a (1,1)-tensor field ϕ, a vector field ξ and 1-form η and a Lorentzian metric g which satisfies the following relations

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$(D_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

and

$$D_X\xi = \phi X, \tag{2.6}$$

for arbitrary vector fields X and Y, where D denotes the covariant differentiation with respect to g [7].

An LP-Sasakian manifold M is said to be Lorentzian Special Para-Sasakian(LSP-Sasakian) manifold if it satisfies

$$F(X,Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.7}$$

where $F(X,Y) = g(\phi X, Y)$ is a symmetric (0,2) tensor.

Let D be the Riemannian connection in M and $\bar{D}$ be an affine connection. Then $\bar{D}$ is said to be metric if it satisfies [7]

$$\bar{D}_X g = 0 \tag{2.8}$$

In an LP-contact manifold M an affine connection $\bar{D}$ is called an ϕ-connection if it satisfies

$$\bar{D}_X\phi = 0 \tag{2.9}$$

Again if

$$\bar{D}_X\xi = \phi X, \tag{2.10}$$

then the connection $\bar{D}$ is said to be semi-symmetric metric ϕ-connection if it satisfies (2.8) and (2.9).

The torsion tensor of connection $\bar{D}$ is of the form

$$T(X,Y) = \eta(Y)\phi(X) - \eta(X)\phi(Y). \tag{2.11}$$

We find the following relations on curvature tensor and Ricci tensor on semi-symmetric metric ϕ-connection on LSP-Sasakian manifold [7].

$$\hat{R}(X,Y)Z = R(X,Y)Z + g(X,Z)Y - g(Y,Z)X, \tag{2.12}$$

$$\hat{S}(X,Y) = S(X,Y) - (n-1)g(X,Y), \tag{2.13}$$

and

$$\hat{r} = r - (n-1), \tag{2.14}$$

where $\hat{R}(X,Y)Z$, $\hat{S}(X,Y)$ and $\hat{r}$ are the curvature tensor, Ricci tensor and scalar curvature respectively with respect to connection $\overline{D}$.

From [1], the pseudo-projective curvature tensor $\tilde{P}(X,Y)Z$, on a n-dimensional Riemannian manifold, is defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] \\ -\frac{r}{n}[\frac{a}{n-1} + b][g(Y,Z)X - g(X,Z)Y], \tag{2.15}$$

Again from [2] we know quasi-conformal curvature tensor on a Riemannian manifold is defined by

$$\tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ -g(X,Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y], \tag{2.16}$$

where a, b are constants and r is the scalar curvature of the Riemannian manifold,

$$g(QX,Y) = S(X,Y) \tag{2.17}$$

and

$$g(\tilde{C}(X,Y)Z,W) = \acute{\tilde{C}}(X,Y,Z,W), \tag{2.18}$$

From [6], the conharmonic curvature tensor on a Riemannian manifold is defined by

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ +g(Y,Z)QX - g(X,Z)QY], \tag{2.19}$$

where

$$g(H(X,Y)Z,W) = \acute{H}(X,Y,Z,W) \tag{2.20}$$

Also from [9] we know the Tachibana concircular curvature tensor on a Riemannian manifold is defined by

$$T(X,Y)Z = R(X,Y)Z - \frac{r}{n(n+2)}\{g(Y,Z)X - g(X,Z)Y + g(\overline{Y},Z)\overline{X} - g(\overline{X},Z)\overline{Y} - 2g(\overline{X},Y)\overline{Z}\}, \tag{2.21}$$

where $\phi X = \overline{X}$.

These results will be required in the next sections.

3. Relation between Pseudo-Projective Curvature Tensor with Semi-Symmetric Metric $\phi-$Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In this section we get the relation between pseudo-projective curvature tensor on a semi-symmetric metric ϕ-connection and pseudo-projective curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold M.

Theorem 3.1 : A necessary and sufficient condition that the pseudo-projective curvature tensor of the semi-symmetric metric ϕ-connection $\bar{D}$ to be equal to pseudo-projective curvature tensor with respect to Levi-Civita connection in an LSP-Sasakian manifold is $a + (n-1)b = 0$.

Proof : Let the pseudo projective curvature tensor of a semi symmetric metric ϕ-connection $\bar{D}$ on LSP-Sasakian manifold is

$$\hat{\tilde{P}}(X,Y)Z = a\hat{R}(X,Y)Z + b[\hat{S}(Y,Z)X - \hat{S}(X,Z)Y] - \frac{\hat{r}}{n}[\frac{a}{n-1} + b][g(Y,Z)X - g(X,Z)Y]. \tag{3.1}$$

Using (2.12), (2.13) and (2.14) in (3.1), we get

$$\hat{\tilde{P}}(X,Y)Z = \tilde{P}(X,Y)Z + a[g(X,Z)Y - g(Y,Z)X] + (n-1)b[g(X,Z)Y - g(Y,Z)X] + \frac{(n-1)}{n}\left[\frac{a}{n-1} + b\right]$$

$$[g(Y,Z)X - g(X,Z)Y],$$

or, $\hat{\tilde{P}}(X,Y)Z = \tilde{P}(X,Y)Z + \dfrac{\{(n-1)b+a\}(n-1)}{n}$

$$[g(X,Z)Y - g(Y,Z)X] \tag{3.2}$$

If $a + (n-1)b = 0$, then $\hat{\tilde{P}}(X,Y)Z = \tilde{P}(X,Y)Z$.

Conversely,

we consider $\hat{\tilde{P}}(X,Y)Z = \tilde{P}(X,Y)Z$, then from (3.2), we get

$$[g(X,Z)Y - g(Y,Z)X]\,\frac{\{(n-1)b+a\}(n-1)}{n} = 0.$$

or, $\quad (1-n)\{a + (n-1)b\} = 0.$

i.e., $\quad a + (n-1)b = 0$, since $n \neq 1$.

4. Relation between Quasi-Conformal Curvature Tensor with Semi-Symmetric Metric $\phi-$Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In this section we get the relation between quasi-conformal curvature tensor on a semi-symmetric metric ϕ-connection and quasi-conformal curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold M.

Theorem 4.1 : A necessary and sufficient condition that the quasi-conformal curvature tensor of the semi-symmetric metric ϕ-connection $\bar{D}$ to be equal to quasi-conformal curvature tensor with respect to Levi-Civita connection in an LSP-Sasakian manifold is $a + (2n-1)b = 0$.

Proof : Taking scalar product of (2.16) and using (2.17) and (2.18), we get

$$\begin{aligned}\acute{C}(X,Y,Z,W) &= a\acute{R}(X,Y,Z,W) + b[S(Y,Z)g(X,W)\\ &-S(X,Z)g(Y,W) + g(Y,Z)S(X,W) - g(X,Z)S(Y,W)]\\ &-\frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)],\end{aligned} \tag{4.1}$$

where

$$\acute{R}(X,Y,Z,W) = g(R(X,Y)Z,W) \tag{4.2}$$

Now we consider quasi-conformal curvature tensor with respect to semi-symmetric metric ϕ-connection $\bar{D}$ on LSP-Sasakian manifold.

$$\begin{aligned}\hat{\acute{\tilde{C}}}(X,Y,Z,W) = a.g(\hat{R}(X,Y)Z,W) + b[\hat{S}(Y,Z)g(X,W) \\ -\hat{S}(X,Z)g(Y,W) + g(Y,Z)\hat{S}(X,W) - g(X,Z)\hat{S}(Y,W)] \\ -\frac{\hat{r}}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{4.3}$$

Using (2.12), (2.13) and (2.14) in (4.3), we obtain

$$\begin{aligned}\hat{\acute{\tilde{C}}}(X,Y,Z,W) = \acute{\tilde{C}}(X,Y,Z,W) + a[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ -b(n-1)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ +g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \frac{(n-1)}{n}\left[\frac{a}{n-1} + b\right] \\ [g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned}$$

$$\begin{aligned}\hat{\acute{\tilde{C}}}(X,Y,Z,W) = \acute{\tilde{C}}(X,Y,Z,W) + [g(X,Z)g(Y,W) - g(Y,Z)g(X,W)] \\ \left[a + 2(n-1)b - \frac{a}{n} - \frac{b(n-1)}{n}\right].\end{aligned} \tag{4.4}$$

If $a + (2n-1)b = 0$, then $\hat{\acute{\tilde{C}}}(X,Y,Z,W) = \acute{\tilde{C}}(X,Y,Z,W)$.

Thus

$$\tilde{C}(X,Y)Z = \hat{\tilde{C}}(X,Y)Z. \tag{4.5}$$

Conversely,

we consider (4.5) holds. Then from (4.4), we get

$$[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)]\left[a + 2(n-1)b - \frac{a}{n} - \frac{b(n-1)}{n}\right] = 0.$$

or, $\quad a + 2(n-1)b - \frac{a}{n} - \dfrac{b(n-1)}{n} = 0.$

or, $\quad (n-1)\{a + b(2n-1)\} = 0.$

Thus $\quad a + b(2n-1) = 0$, since $n \neq 1$.

5. Relation between Conharmonic Curvature Tensor with Semi- Symmetric Metric $\phi-$Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In present section we get the relation between conharmonic curvature tensor on a semi-symmetric metric ϕ-connection and conharmonic curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold M.

Theorem 5.1 : In an LSP-Sasakian manifold the conharmonic curvature tensor $\hat{\acute{H}}(X,Y,Z,W)$ of a semi symmetric metric ϕ-connection $\bar{D}$ is

$$\hat{\acute{H}}(X,Y,Z,W) = \acute{H}(X,Y,Z,W) + k[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)],$$

where $k = [1 + \frac{2(n-1)}{n-2}]$.

Proof : Taking scalar product of (2.19) and using (2.17), (2.20) and (4.2),one can obtain

$$\acute{H}(X,Y,Z,W) = \acute{R}(X,Y,Z,W) - \tfrac{1}{n-2}[S(Y,Z)g(X,W)$$
$$-S(X,Z)g(Y,W) + S(X,W)g(Y,Z) - S(Y,W)g(X,Z)]$$

Now we consider the conharmonic curvature tensor of a semi symmetric metric ϕ-connection $\bar{D}$ on LSP-Sasakian manifold is

$$\hat{\acute{H}}(X,Y,Z,W) = \hat{\acute{R}}(X,Y,Z,W) - \tfrac{1}{n-2}[\hat{S}(Y,Z)g(X,W)$$
$$-\hat{S}(X,Z)g(Y,W) + \hat{S}(X,W)g(Y,Z) - \hat{S}(Y,W)g(X,Z)]$$

Using (2.12) and (2.13), we get

$$\hat{\acute{H}}(X,Y,Z,W) = \acute{H}(X,Y,Z,W) + g(X,Z)g(Y,W) - g(Y,Z)g(X,W)$$
$$-\frac{2(n-1)}{n-2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)$$
$$= \acute{H}(X,Y,Z,W) + [g(X,Z)g(Y,W)$$
$$-g(Y,Z)g(X,W)][1 + \frac{2(n-1)}{n-2}].$$

$$\hat{\acute{H}}(X,Y,Z,W) = \acute{H}(X,Y,Z,W) + k[g(X,Z)g(Y,W) - g(Y,Z)g(X,W)],$$

where $k = [1 + \frac{2(n-1)}{n-2}]$.

6. Relation between Tachibana Concircular Curvature Tensor with Semi-Symmetric Metric $\phi-$Connection and with Levi-Civita Connection on LSP-Sasakian Manifold

In this section we get the relation between Tachibana concircular curvature tensor on a semi-symmetric metric ϕ-connection and Tachibana concircular curvature tensor on Levi-Civita connection in an LSP-Sasakian manifold M.

Theorem 6.1 : In an LSP-Sasakian manifold the Tachibana concircular curvature tensor $\hat{T}(X,Y)Z$ of a semi symmetric metric ϕ-connection $\bar{D}$ is

$$\hat{T}(X,Y)Z = T(X,Y)Z + l[g(X,Z)Y - g(Y,Z)X] + m\{g(\overline{Y},Z)\overline{X} - g(\overline{X},Z)\overline{Y} \\ -2g(\overline{X},Y)\overline{Z}\},$$

where $l = 1 - \frac{n-1}{n(n-2)}$ and $m = \frac{n-1}{n(n-2)}$.

Proof : Let the Tachibana concircular curvature tensor of a semi symmetric metric ϕ-connection $\bar{D}$ on LSP-Sasakian manifold is

$$\hat{T}(X,Y)Z = \hat{R}(X,Y)Z - \frac{\hat{r}}{n(n+2)}\{g(Y,Z)X - g(X,Z)Y + g(\overline{Y},Z)\overline{X} \\ -g(\overline{X},Z)\overline{Y} - 2g(\overline{X},Y)\overline{Z}\}.$$

Using (2.12) and (2.13), we obtain

$$\begin{aligned}\hat{T}(X,Y)Z &= T(X,Y)Z + g(X,Z)Y - g(Y,Z)X + \frac{(n-1)}{n(n+2)}\{g(Y,Z)X \\ &\quad -g(X,Z)Y + g(\overline{Y},Z)\overline{X} - g(\overline{X},Z)\overline{Y} - 2g(\overline{X},Y)\overline{Z}\}. \\ &= T(X,Y)Z + [g(X,Z)Y - g(Y,Z)X]\left[1 - \frac{n-1}{n(n-2)}\right] \\ &\quad + \frac{n-1}{n(n-2)}\{g(\overline{Y},Z)\overline{X} - g(\overline{X},Z)\overline{Y} - 2g(\overline{X},Y)\overline{Z}\}.\end{aligned}$$

or, $$\hat{T}(X,Y)Z = T(X,Y)Z + l[g(X,Z)Y - g(Y,Z)X] \\ +m\{g(\overline{Y},Z)\overline{X} - g(\overline{X},Z)\overline{Y} - 2g(\overline{X},Y)\overline{Z}\},$$

where $l = 1 - \frac{n-1}{n(n-2)}$ and $m = \frac{n-1}{n(n-2)}$.

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