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Riccion Field from Higher-Dimensional Theory and Renormalization

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Abstract

Riccion fields are obtained through spontaneous compactification of $(4+D)-$dimensional Kaluza-Klein type theory. It is found that multiplicatively renormalizable quantum theory free from non-unitarity problem can be obtained for Riccion fields which manifest material aspect of Recci scalar. The renormalization group improved effective lagrangian for these fields is also derived.

Keywords : Higher-dimensional and higher -derivative gravity; renormalization; lagrangian density; quantum theory.
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1. Introduction

It is believed, in Kaluza-Klein type theories that, at extremely high energy levels (above Planck scale), the space-time need not be four-dimensional as it is observed. These theories are higher-dimensional, where space-time is taken to be $(4+D)-$dimensional. As observed universe is 4-dimensional, it is also believed that spontaneous compactification takes place at energy below Planck scale [1]. The topology of the space-time is taken as $M^4 \times B^D$, where M^4 is the usual 4-dimensional space-time and B^D is the D-dimensional compact space. Here, it is planned to take B^D as T^D (D-dimensional torus).

The line-element for $(4+D)-$dimensional space-time with topology $M^D \times T^D$ is taken as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \rho_1^2 d\theta_1^2 + \rho_2^2 d\theta_2^2 + \cdots + \rho_D^2 d\theta_D^2 \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$; T^D is the product of D-copies of circles with different radii $\rho_1, \rho_2, \ldots, \rho_D$ and $0 \leq \theta_1, \theta_2, \ldots, \theta_D \leq 2\pi$.

In the present paper, a scalar field ψ is considered in the space-time given by equation (1). After spontaneous compactification, a four-dimensional action for the scalar field is obtained. One-loop correction to the four-dimensional scalar induces higher-derivative gravity terms. As a result, an action for higher-derivative gravity is obtained.

In what follows, trace of the resulting gravitational field equations are obtained exhibiting matter aspect of Ricci scalar R manifested through a scalar field $\tilde{\phi} = \eta R$ (η is a constant of length dimension and unit magnitude) called Riccion [2]. Here, one-loop correction is done to Riccion-field using operator regularization method [3] and renormalization group improved effective potential for $\tilde{\phi}$ is derived.

Natural units ($h = c = k_B = 1$, where h, c and k_B have their usual meaning) are used throughout the paper. M_P stands for planck mass.

2. Spontaneous compactification and Riccion-field

In $(4 + D)-$dimensional space-time, the action for gravity and scalar field ψ is taken as

$$S = S_g + S_\psi = \int d^4x d^D y \sqrt{|g_{4+D}|}$$

$$\left[-\frac{R_{4+D}}{16\pi G_{4+D}} + \frac{1}{2}\{g^{MN}(D_M\psi^*)(D_N\psi) - (\xi' R_{4+D} + m_0^2)\psi^*\psi\} \right] \tag{2}$$

where x^μ are co-ordinates of M^4, $y_1 = \rho_1\theta_1$, $y_2 = \rho_2\theta_2$, $\ldots$, $y_D = \rho_D\theta_D$, g_{4+D} is the determinant of the metric tensor, R_{4+D} is the Ricci scalar in $M^4 \times T^D$, $D_\mu = \nabla_\mu$ are covariant derivatives in M^4, $D_{m'} = \partial_{m'}$ ($m' = 1, 2, \ldots, D$), $M = (\mu, m')$, m_o is the mass of ψ field and G_{4+D} is the (4+D)-dimensional gravitational constant.

In the space-time with topology $M^4 \otimes T^D$, $\psi(x, y)$ can be decomposed as

$$\psi(x, y) = [(2\pi)^D \rho_1\rho_2 \ldots \rho_D]^{-1/2} \times \sum_{n_1 \ldots n_D = -\infty}^{\infty} \psi_{(n)}(x) \exp\left[i \sum_{j=1}^{D} \frac{2\pi(n_j + \alpha)}{\rho_j} y_j \right] \tag{3}$$

where $\alpha = o(\frac{1}{2})$ for untwisted (twisted) fields. As T^D is not simply connected manifold. So, possibility exists for both twisted and untwisted fields on T^D, here $\alpha = 0$ is taken. Henceforth, as untwisted fields are abundant in the nature.

So, the result is obtained from equations (2) and (3) as

$$S_{\psi}^{(4)} = -\frac{1}{2}\int d^4x\sqrt{-g_4} \sum_{n_1 \ldots n_D=-\infty}^{\infty} \psi^*_{(n)}(\Box_4 + m^2_{(n)})\psi_{(n)} \tag{4a}$$

where $\psi_{(n)} = \psi_{n_1 n_2 \ldots n_D}$ and

$$m^2_{(n)} = m_o^2 + \xi' R_4 + (2\pi)^2\left(\frac{n_1^2}{\rho_1^2} + \cdots + \frac{n_D^1}{\rho_D^2}\right) \tag{4b}$$

Also from equation (2), one obtains through spontaneous compatification on T^D

$$S_g^{(4)} = -\frac{1}{16\pi G_4}\int d^4x\sqrt{-g_4}R_4 \tag{5}$$

where $G_4 = G_{4+D}/(2\pi)^D \rho_1\rho_2 \ldots \rho_D$.

For one-loop quantum correction to scalar fields $\psi_{(n)}$, operator regularization method is used which is very convenient for regularization in 4-dimensional curved spaces [3]. Using this method, one-loop effective action for $\psi_{(n)}$ is obtained upto adiabatic order 4 as

$$\begin{aligned}\Gamma =& S_\psi^4 + \sum_{n_1 \ldots n_D=-\infty}^{\infty} \frac{d}{ds}\{(\frac{\mu^2}{m_n^2})^s \int d^4x\sqrt{-g_4} \\ &\times [\frac{m_n^2}{(s-2)(s-1)} + \frac{m_n^2}{(s-1)}(\frac{1}{6}-\xi')R \\ &+ \{\frac{1}{6}(\frac{1}{5}-\xi')\Box R + \frac{1}{180}(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu}) \\ &+ \frac{1}{2}(\xi' - \frac{1}{6})^2 R^2\}]|s=0\end{aligned} \tag{6}$$

The summation in equation (6), can be done using the procedure adapted in ref. [2]. Here, onwards suffix 4 is dropped, as further analysis is confined to M^4 only. As a result, equation (6) is re-written as

$$\begin{aligned}\Gamma =& S_\psi^{(4)} + \frac{1}{16\pi^2}\int d^4x\sqrt{-g_4}[\ln(\mu^2/m_o^2 + \xi' R)\{\frac{1}{2}(m_o^2+\xi' R)^2 + \\ &(m_o^2+\xi' R)(\xi' - \frac{1}{6})R + \frac{1}{6}(\frac{1}{5}-\xi')\Box R + \frac{1}{180}(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu} + \\ &\frac{1}{2}(\xi' - \frac{1}{6})^2R^2\} - \frac{3}{4}(m_o^2+\xi' R)^2 - (m_o^2+\xi' R)(\frac{1}{6}-\xi')R]\end{aligned} \tag{7}$$

$(4+D)-$dimensional space-time reduces to 4-dimensional one at or below Plank scale. At these scales, terms containing derivatives of higher order than R^3, $R\Box R$ and $R(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu})$ can be conveniently ignored. Now expanding the logarithmic term and taking $\mu^2 = m_o^2$, one gets from equation (5) and (7)

$$S_{g^{(4)}} + \Gamma = S_\psi^{(4)} - \frac{1}{16\pi}\int d^4x\sqrt{-g_4}[\frac{R}{G} + \frac{3\xi'^2R^2}{2\pi} + \frac{\xi'}{6\pi m_o^2}(\xi'^2 + \frac{1}{2}\xi' - \frac{1}{12})R^3 - \frac{\xi'}{30\pi m_o^2}(5\xi'-1)R\Box R + R\frac{\xi'}{180\pi m_o^2}(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu})] \quad (8)$$

neglecting $3m_o^4/4$ and taking approximation $\frac{1}{G} + \frac{m_o^2}{2}(\frac{1}{3}+\xi') \approx \frac{1}{G}$. It is physically reasonable to treat m_o sufficiently small as it is mass of ψ-field in higher-dimensional space-time.

Invariance of the action, given by equation (8) under transformation $g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu}$ leads to fields equations [2,4,5].

$$G^{-1}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \frac{3\xi'^2}{2\pi}(2R;_{\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}) + \frac{\xi'}{6\pi m_o^2}(\xi'^2 + \frac{1}{2}\xi' - \frac{1}{12})(3R^2R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^3 + 6R^2;_{\mu\nu} - 6g_{\mu\nu}\Box R^2) + \xi'\frac{1}{180\pi m_o^2}R(-\frac{1}{2}g_{\mu\nu}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + 2R_{\mu\alpha\beta\gamma}R_\nu^{\alpha\beta\gamma} - 4\Box R_{\mu\nu} + 2R;_{\mu\nu} - 4R_{\mu\alpha}R_\nu^\alpha + 4R^{\alpha\beta}R_{\alpha\mu\beta\nu} - 2R^\alpha_{\mu;\nu\alpha} + \Box R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\Box R - 2R_\mu^\alpha R_{\alpha\nu} + \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}) + \frac{\xi'}{180\pi m_o^2}\{(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta});\mu\nu - \Box(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta})\} - 8\pi\langle T_{\mu\nu}\rangle = 0 \quad (9)$$

where semi-colon (;) denotes covariant derivative in curved space-time, and

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\right).$$

Vacuum expectation value of components of energy-momentum tensor is given as $\langle T_{\mu\nu}\rangle$ here.

Trace of equation (9) is given as

$$\begin{aligned}&\Box R + (\frac{\pi}{9\xi'^2 G})R - (\frac{1}{18\xi' m_o^2})(\xi'^2 + \frac{1}{2}\xi' - \frac{1}{12})R^3 + (\frac{1}{3\xi' m_o^2})\\ &[\Box R^2 + (\frac{1}{180})\Box(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta})] + (\frac{8\pi^2}{9\xi'^2})\langle T\rangle = 0\end{aligned} \qquad (10)$$

As dynamical contribution of terms

$$\int d^4x\sqrt{-g}\Box R^2 \text{ and } \int d^4x\sqrt{-g}(\frac{1}{180})\Box(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta})$$

vanishes, so

$$g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}\Box R^2 = 0.$$

and

$$g^{\mu\nu}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}\Box(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta}) = 0.$$

These equations imply that

$$\Box R^2 = 0 = \Box(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta}) \qquad (11a, b)$$

Trace of components of energy-momentum tensor obtained from $S_\psi^{(4)}$, given by equation (4) is

$$T = \sum_{n_1 \ldots n_D = -\infty}^{\infty} m_{(n)}^2 \psi_{(n)}^* \psi_{(n)}$$

which yields

$$\langle T\rangle \propto (m_0^2 + \xi' R) \qquad (12)$$

performing summation as above. So $\langle T\rangle$ can be neglected compared to other geometric terms in equation (10). Thus from equations (10) and (11), one obtains

$$\Box\tilde{\phi} + m^2\tilde{\phi} + \frac{\lambda}{6}\tilde{\phi}^3 = 0 \qquad (13)$$

where

$$\tilde{\phi} = \eta R,$$

$$m^2 = (\frac{\pi}{9\xi'^2 G})$$

and

$$\lambda = -(\frac{1}{3\xi' m_o^2\eta^2})(\xi'^2 + \frac{1}{2}\xi' - \frac{1}{12}).$$

If $G = G_N$ (the Newtonian gravitational constant), for $\xi' > 0.59$, $m < M_P$ and for $\xi' = 0.6$, $m = 9.85 \times 10^{18}$Gev as $G_N \simeq M_\rho^{-2}$. The equation (13) is the semi-classical equation for $\tilde{\phi}$ in curved space-time. The lagrangian density leading to this equation is given as

$$\mathcal{L} = \frac{1}{2}(g^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi} - m^2\tilde{\phi}^2) - \frac{\lambda}{4!}\tilde{\phi}^4 \tag{14a}$$

with the action

$$S_{\tilde{\phi}} = \int d^4x\sqrt{-g}\mathcal{L}.$$

In equations (13) and (14), $m^2 > 0$ and Riccion field $\tilde{\phi}$ has self-interaction term. So $\tilde{\phi}$ is free from the tachyon ghost problem. Equation (13) can be obtained from equations (14) also using invariance of $S_{\tilde{\phi}}$ under transformation $\tilde{\phi} \longrightarrow \tilde{\phi} + \delta\tilde{\phi}$. $\tilde{\phi}$ is different from other spinless matter fields in the sense that at the classical level,

$$\tilde{\phi}_{classical} = \hat{\tilde{\phi}} = \eta R = \eta[g^{\mu\nu}(\Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu}) + g^{\mu\nu}(\Gamma^\alpha_{\alpha\beta}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu})] \tag{15}$$

where $\Gamma^\alpha_{\mu\nu}$ are affine connections in Riemannian geometry defined as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta})$$

and comma (,) denotes partial derivatives.

3. One-loop correction to $\tilde{\phi}$, renormalization and renormalization group improved potential for $\tilde{\phi}$.

The one-loop correction to $\tilde{\phi}$, is given as

$$\tilde{\Gamma}^{(1)} = \frac{i}{2}\ln Det(D/\mu^2) \tag{16a}$$

where

$$D = \Box + m^2 + \frac{\lambda}{2}\hat{\tilde{\phi}}^2. \tag{16b}$$

Here, $\hat{\tilde{\phi}}$ is the classical minimum of the quantum field $\tilde{\phi}$ with fluctuation $\tilde{\phi} - \hat{\tilde{\phi}}$. $\tilde{\Gamma}^{(1)}$ can be evaluated using the operator regularization method (used above) up to adiabatic order 4 as,

$$\tilde{\Gamma}^{(1)} = \frac{1}{16\pi^2}\int d^4x\sqrt{-g}[(m^2 + \frac{\lambda}{2}\hat{\tilde{\phi}}^2)^2\{\frac{3}{4} - \frac{1}{2}\ln(\frac{m^2 + \frac{\lambda}{2}\hat{\tilde{\phi}}^2}{\tilde{\mu}^2})\}$$
$$- \frac{1}{6}R(m^2 + \frac{\lambda}{2}\hat{\tilde{\phi}}^2)\{1 - \ln(\frac{m^2 + \frac{\lambda}{2}\hat{\tilde{\phi}}^2}{\tilde{\mu}^2})\} - \ln(\frac{m^2 + \frac{\lambda}{2}\hat{\tilde{\phi}}^2}{\tilde{\mu}^2})$$
$$\{\frac{1}{30}\Box R + \frac{1}{72}R^2 + \frac{1}{180}(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} - R^{\mu\nu}R_{\mu\nu})\}] \tag{17}$$

In the operator regularization method, normal co-ordinates are used. $\hat{\tilde{\phi}}$, being a scalar field, remains invariant in these co-ordinates.

The renormalized form of one-loop effective lagrangian density for $\tilde{\phi}$ can be written, using euations (14) and (17) as

$$\mathcal{L}_{\rm ren} = \frac{1}{2}(g^{\mu\nu}\partial_\mu\hat{\tilde{\phi}}\partial_\nu\hat{\tilde{\phi}} - m^2\hat{\tilde{\phi}}^2) - \frac{\lambda}{4!}\hat{\tilde{\phi}}^4 + \wedge + \in_0 R + \frac{1}{2}\in_1 R^2 +$$
$$\in_2 R^{\mu\nu}R_{\mu\nu} + \in_3 R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + \in_4 \Box R + \tilde{\Gamma}^{(1)} + \mathcal{L}_{ct} \tag{18a}$$

where $\mathcal{L}_{ct}$ is the counter-term contribution given as

$$\mathcal{L}_{ct} = -\frac{1}{2}\delta m^2\hat{\tilde{\phi}}^2 - \frac{1}{4!}\delta\lambda\hat{\tilde{\phi}}^4 + \delta\wedge - \frac{1}{2}\delta\xi R\hat{\tilde{\phi}}^2 + \delta\in_0 R + \frac{1}{2}\delta\in_1 R^2 +$$
$$\delta\in_2 R^{\mu\nu}R_{\mu\nu} + \delta\in_3 R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + \delta\in_4 \Box R \tag{18b}$$

In equations (18) ξ, $\in_0, \in_1$, $\in_2$, $\in_3$, and $\in_4$ are dimensionless coupling constants. Counter-terms δm^2, $\delta\lambda$, $\delta\wedge$, $\delta\xi$, $\delta\in_0$, $\delta\in_1$, $\delta\in_2$, $\delta\in_3$ and $\delta\in_4$ can be evaluated using the following renormalization conditions [6]

$$\wedge = \mathcal{L}_{\rm ren}|\hat{\tilde{\phi}} = \phi_0,\ R = 0$$
$$\lambda = -\frac{\partial^4\mathcal{L}_{\rm ren}}{\partial\hat{\tilde{\phi}}^4}|\hat{\tilde{\phi}} = \phi_1,\ R = 0$$
$$m^2 = -\frac{\partial^2\mathcal{L}_{\rm ren}}{\partial\hat{\tilde{\phi}}^2}|\hat{\tilde{\phi}} = 0\ , R = 0$$
$$\xi = -\frac{\partial^3\mathcal{L}_{\rm ren}}{\partial R\partial\hat{\tilde{\phi}}^2}|R = 0,\ \hat{\tilde{\phi}} = \phi_3$$
$$\in_0 = \frac{\partial\mathcal{L}_{\rm ren}}{\partial R}|\hat{\tilde{\phi}} = 0,\ R = 0, \tag{19}$$

$$\begin{aligned}
\in_0 &= \frac{\partial^2 \mathcal{L}_{\text{ren}}}{\partial R^2} |\hat{\tilde{\phi}} = 0, \;\; R = R_5, \\
\in_2 &= \frac{\partial \mathcal{L}_{\text{ren}}}{\partial (R^{\mu\nu} R_{\mu\nu})} |\hat{\tilde{\phi}} = 0, \;\; R = R_6, \\
\in_3 &= \frac{\partial \mathcal{L}_{\text{ren}}}{\partial (R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta})} |\hat{\tilde{\phi}} = 0, \;\; R = R_7, \\
\in_4 &= \frac{\partial \mathcal{L}_{\text{ren}}}{\partial \Box R} |\hat{\tilde{\phi}} = 0, \;\; R = R_8.
\end{aligned}$$

Since $\tilde{\phi} = \eta R$, so when $R = 0, \phi_0 = \phi_1 = \phi_3 = 0$. Similarly when $\hat{\tilde{\phi}} = 0$, $R_5 = R_6 = R_7 = R_8 = 0$.

The counter-terms obtained are given as

$$\begin{aligned}
16\pi^2 \delta \wedge &= -m^4 [\frac{3}{4} - \frac{1}{2} \ln(m^2/\tilde{\mu}^2)] \\
16\pi^2 \delta \lambda &= -3\lambda^2 [\frac{3}{2} - \ln(m^2/\tilde{\mu}^2)] \\
16\pi^2 \delta m^2 &= 2\lambda m^2 [1 - \ln(m^2/\tilde{\mu}^2)] \\
96\pi^2 \delta \xi &= -\lambda \ln(m^2/\tilde{\mu}^2) \\
96\pi^2 \delta \in_0 &= m^2 [1 - \ln(m^2/\tilde{\mu}^2)] \\
1152\pi^2 \delta \in_1 &= \ln(m^2/\tilde{\mu}^2) \\
2880\pi^2 \delta \in_2 &= -\ln(m^2/\tilde{\mu}^2) \\
2880\pi^2 \delta \in_3 &= \ln(m^2/\tilde{\mu}^2) \\
480\pi^2 \delta \in_4 &= \ln(m^2/\tilde{\mu}^2).
\end{aligned} \tag{20}$$

Using equations (20) in equations (18), one obtains

$$\begin{aligned}
\mathcal{L}_{\text{ren}} =& \frac{1}{2}(g^{\mu\nu} \partial_\mu \hat{\tilde{\phi}} \partial_\nu \hat{\tilde{\phi}} - m^2 / \hat{\tilde{\phi}}^2) - \\
& \frac{\lambda}{4!} \hat{\tilde{\phi}}^4 + \wedge + \in_0 R + \frac{1}{2} \in_1 R^2 + \in_2 R^{\mu\nu} R_{\mu\nu} + \\
& \in_3 R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} + \in_4 \Box R - 16\pi^2 \ln(1 + \frac{\lambda \hat{\tilde{\phi}}^2}{2m^2}) \\
& [\frac{1}{2}(m^2 + \frac{\lambda}{2} \hat{\tilde{\phi}}^2)^2 - \frac{1}{6} R (m^2 + \frac{\lambda}{2} \hat{\tilde{\phi}}^2) + \\
& \frac{1}{30} \Box R + \frac{1}{72} R^2 + \frac{1}{180} (R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} - R^{\mu\nu} R_{\mu\nu})]
\end{aligned} \tag{21}$$

The effective renormalized lagrangian can be improved further solving renormalization group equations. The corresponding β- functions are calculated using counter- terms given by equations (20). The resulting renormalization group equations are obtained as [6-8]

$$\begin{aligned}
\frac{d\wedge}{dt} &= \frac{m^4}{32\pi^2} \\
\frac{d\lambda}{dt} &= -\frac{3\lambda^3}{16\pi^2} \\
\frac{dm^2}{dt} &= -\frac{\lambda m^2}{32\pi^2} \\
\frac{d\in_0}{dt} &= -\frac{m^2}{96\pi^2} \\
\frac{d\in_1}{dt} &= \frac{1}{1152\pi^2}\ln(m^2/m_c^2) \\
\frac{d\in_2}{dt} &= -\frac{1}{2880\pi^2}\ln(m^2/m_c^2) \\
\frac{d\in_3}{dt} &= \frac{1}{2880\pi^2}\ln(m^2/m_c^2) \\
\frac{d\in_4}{dt} &= \frac{1}{480\pi^2}\ln(m^2/m_c^2)
\end{aligned} \tag{22}$$

where $t = \ln(m_c^2/\tilde{\mu}^2)$ with m_c as cut-off mass-scale. Equations (22) yield solutions

$$\begin{aligned}
\wedge &= \wedge_0 + \frac{m_0^4}{2\lambda_0}[(1+\frac{3\lambda_0 t}{16\pi^2})^{\frac{1}{3}} - 1] \\
\lambda &= \lambda_0(1+\frac{3\lambda_0 t}{16\pi^2})^{-1} \\
m^2 &= m_0^2(1+\frac{3\lambda_0 t}{16\pi^2})^{-\frac{1}{3}} \\
\in_0 &= \in_{00} + 8\pi^2 m_0^2[1-(1+\frac{3\lambda_0 t}{16\pi^2})^{2/3}] \\
\in_1 &= \in_{10} + \frac{t}{1152\pi^2}\ln(m^2/m_c^2) \\
\in_2 &= \in_{20} - \frac{t}{2880\pi^2}\ln(m^2/m_c^2) \\
\in_3 &= \in_{30} + \frac{t}{2880\pi^2}\ln(m^2/m_c^2) \\
\in_4 &= \in_{40} + \frac{t}{480\pi^2}\ln(m^2/m_c^2)
\end{aligned} \tag{23}$$

where $\wedge_0,\ \lambda_0,\ m_0^2,\ \in_{00},\ \in_{10},\ \in_{20},\ \in_{30}$ and $\in_{40}$ are coupling constants evaluated at $t = 0$.

4. Conclusion

Using the definition of λ given by equation (13), in the equation (2), it is obtained that

$$(\xi' + \frac{1}{2} - \frac{1}{12\xi'}) = (\xi'_0 + \frac{1}{2} - \frac{1}{12\xi'_0}) \left[1 - \frac{(\xi'_0 + \frac{1}{2} - \frac{1}{12\xi'_0})t}{16\pi^2 m_0 \eta^2} \right]^{-1} \tag{24}$$

which implies that

$$\xi' = (\xi'_0 + \frac{1}{2}) \left[1 - \frac{(\xi_0 + \frac{1}{2})t}{16\pi^2 m_0^2 \eta^2} \right]^{-1} - \frac{1}{2} \tag{25a}$$

in case $\xi'^2 + \frac{\xi'}{2} > \frac{1}{12}$ and

$$\xi' = \xi'_0 + \frac{t}{192\pi^2 m_0^2 \eta^2} \tag{25b}$$

in case $\xi' + \frac{\xi'}{2} < \frac{1}{12}$.

The definition of m^2, from equation (13), with the help of equation (23) yields that

$$\xi'^2 G = \xi'_0 G_0 (1 + \frac{3\lambda_0 t}{16\pi^2})^{\frac{1}{2}} \tag{26}$$

If $\xi'^2 + \frac{\xi'}{2} > \frac{1}{12}$, equation (26) implies that

$$G = \frac{\xi_0^2 G_0 [1 - (\xi'_0 + \frac{1}{2})t/16\pi^2 m_0^2 \eta^2]^{\frac{7}{3}}}{[(\xi'_0 + \frac{1}{2})(1 + t/16\pi^2 m_0^2 \eta^2) - \frac{1}{2}]^2} \tag{27a}$$

and

$$G = G_0 (1 + \frac{t}{192\pi^2 m_0^2 \xi'_0 \eta^2})^{-5/3} \tag{27b}$$

if $\xi'^2 + \frac{\xi'}{2} < \frac{1}{12}$. Equation (27a) implies that $G \longrightarrow \infty$, as $\tilde{\mu}^2 \longrightarrow \infty$, whereas equation (27b) shows that $G \longrightarrow 0$ as $\tilde{\mu}^2 \longrightarrow \infty$.

Thus, it is found that we can get a multiplicatively renormalizable quantum theory for $\tilde{\phi} = \eta R$, at high energy level which is free from non-unitarity problem.

Moreover, solutions of renormalization group equations, given by equations(23), show that curvature terms are very strong at high energy.

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