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On Special Curvature Tensor in a Generalized 2-recurrent Smooth Riemannian Manifold

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Abstract

In this paper, we have discussed on a special type of curvature tensor in a smooth Riemannian manifold and have studied its cyclic and differentiable properties. We have also studied the 2-recurrence properties of the tensor S, T, F and H in Riemannian manifold as well as in an Einstein manifold.

1. Introduction

Special curvature tensor has been introdued and studied by Singh and Khan [6]. Let M_n be an $n-$dimensional smooth Riemannian manifold and X, Y, Z and W be differentiable vector field on M_n. A special curvature tensor $H(X, Y, Z)$ of type (1, 3) has been defined as [6] :

$$H(X, Y, Z) = R(X, Y, Z) + R(X, Z, Y), \tag{1.1}$$

$$< H(X, Y, Z), W > = < R(X, Y, Z), W > + < R(X, Z, Y), W >, \tag{1.2}$$

or

$$'H(X, Y, Z, W) = {}'R(X, Y, Z, W) + {}'R(X, Z, Y, W). \tag{1.3}$$

It is obvious that

$$H(X, Y, Z) = H(X, Z, Y),$$

which shows that it is symmetric in last two slots. Sinha [5] has defined and studied certain tensors of type (1, 3) in a smooth Riemannian manifold. They are

$$S(X, Y, Z) = Ric(Y, Z)\, X + Ric(Z, X)\, Y + Ric(X, Y)\, Z, \tag{1.4}$$

$$T(X, Y, Z) = < Y, Z > \ X + < Z, X > \ Y + < X, Y > \ Z, \tag{1.5}$$

$$F(X, Y, Z) = < Y, Z > \ K(X) + < Z, X > \ K(Y) + < X, Y > \ K(Z), \tag{1.6}$$

which are symmetric in X, Y, Z.

A special curvature tensor $H(X,Y,Z)$ has cyclic property [4].

$$H(X,Y,Z)+H(Y,Z,X)+H(Z,X,Y)=0. \tag{1.7}$$

In 1972 A. K. Roy generalized the notion of 2-recurrent manifold. A Riemannian manifold (M^n,g) is called generalized 2-recurrent, if the Riemannian curvature tensor satisfies the condition

$$(D_V D_U R)(X,Y)Z = A(V)(D_U R)(X,Y,Z)+B(U,V)R(X,Y,Z) \tag{1.8}$$

where A is non-zero 1-form and B is non-zero 2-form tensor. D denote the covariant differentiation with respect to metric tensor.

In a recent paper, De and Bandyopadhyay [2] introduced and studied generalized Ricci 2-recurrent Riemannian manifold which are defined as : A non-flat Riemannian manifold is called generalized Ricci 2-recurrent Riemannian manifold if the Ricci tensor is non-zero and satisfies the condition

$$(D_V D_U Ric)(X,Y)Z = A(V)(D_U Ric)(X,Y)+B(U,V)Ric(X,Y) \tag{1.9}$$

where A and B are stated earlier. If the 2-form $B(U,V)$ becomes zero, then the space reduces to Ricci recurrent space.

An $n-$dimensional smooth Riemannian manifold M_n is called an Einstein manifold, if for all $X,Y \in \chi(M_n)$

$$Ric(X,Y) = k < X,Y >, \tag{1.10}$$

where $k : M_n \to R$ real is a valued function.

In this paper, we have studied some theorems about special curvature tensor $H(X,Y,Z)$. In section two of this paper, we have studied the 2-recurrent properties of the tensors S,T,F and H in a smooth Riemannian manifold as well as in an Einstein manifold. In section third of this paper, we have studied its cyclic and bi-covariant differentiation properties in generalized 2-recurrent smooth Riemannian manifold.

2. Recurrence Properties of (1, 3) type Tensors in a Generalized 2-recurrent Smooth Riemannian Manifold

Let M_n be an $n-$dimensional smooth Riemannian manifold. Then, M_n is called generalized 2-recurrent smooth Riemannian manifold with respect to the tensor $H(X,Y,Z)$, if

$$(D_V D_U H)(X,Y,Z) = A(V)(D_U H)(X,Y,Z)+B(U,V)H(X,Y,Z), \tag{2.1}$$

where A is $1-$form and B is $2-$form known as recurrence parameter. A smooth Riemannian manifold M_n is called $2-$recurrent with respect to tensor $S(X, Y, Z)$, $T(X, Y, Z)$ and $F(X, Y, Z)$ defined by equations (1.4), (1.5) and (1.6) respectively, if

$$(D_V D_U Q)(X, Y, Z) = A(V)(D_U Q)(X, Y, Z) + B(U, V)Q(X, Y, Z), \qquad (2.2)$$

where A and B are stated earlier and Q stands for S, T and F, respectively. We now prove the following :

Theorem (2.1). An $n-$dimensional smooth Riemannian manifold M_n is generalized $2-recurrent$ with respect to the tensor $H(X, Y, Z)$, if it is a generalized 2-recurrent smooth Riemannian manifold with the same recurrence parameter.

Proof. Taking bi-covariant derivative of equation (1.1) with respect to 'U' and 'V', we get

$$(D_V D_U H)(X, Y, Z) = (D_V D_U R)(X, Y, Z) + (D_V D_U R)(X, Z, Y). \qquad (2.3)$$

On using equation (1.8) in equation (2.3), we get

$$\begin{aligned}(D_V D_U H)(X, Y, Z) &= A(V)(D_U R)(X, Y, Z) + B(U, V)R(X, Y, Z)\\ &+A(V)(D_U R)(X, Z, Y) + B(U, V)R(X, Z, Y). \qquad (2.4)\end{aligned}$$

On using equation (1.1) in equation (2.4), we get

$$(D_V D_U H)(X, Y, Z) = A(V)(D_U H)(X, Y, Z) + B(U, V)H(X, Y, Z). \qquad (2.5)$$

That is, M_n is generalized $2-$recurrent with respect to tensor $H(X, Y, Z)$.

Theorem (2.2). If a smooth Riemannian manifold M_n is generalized $2-$recurrent with respect to the special tensor $H(X, Y, Z)$, then

$$\begin{aligned}A(V)(D_U H)(X, Y, Z) + B(U, V)H(X, Y, Z) &= (D_U D_V R)(X, Y, Z)\\ &+(D_U D_V R)(X, Z, Y). \qquad (2.6)\end{aligned}$$

Proof. Let M_n be 2-recurrent Riemannian manifold with respect to the tensor $H(X, Y, Z)$, then from equation (2.1), we have

$$\begin{aligned}A(V)\{(D_U R)(X, Y, Z) + (D_U R)(X, Z, Y) + B(U, V)\{R(X, Y, Z) + R(X, Z, Y)\}\\ = (D_U D_V R)(X, Y, Z) + (D_U D_V R)(X, Z, Y), \qquad (2.7)\end{aligned}$$

$$\begin{aligned}(D_U D_V R)(X, Y, Z) - A(V)(D_U R)(X, Y, Z) - B(U, V)R(X, Y, Z)\\ +(D_U D_V R)(X, Z, Y) = -A(V)(D_U R)(X, Z, Y) - B(U, V)R(X, Z, Y) = 0. \qquad (2.8)\end{aligned}$$

Since M_n is generalized 2-recurrent with respect to tensor $H(X,Y,Z)$. Therefore, on using equations (1.8) and (1.1) in equation (2.8), we get the required result.

Theorem (2.3). An Einstein manifold M_n is generalized 2−recurrent with respect to the tensor $T(X,Y,Z)$, if it is generalized Ricci 2-recurrent for the same recurrence 2-form.

Proof. On using equation (1.5) in equation (2.8), we get

$$T(X,Y,Z) = \frac{1}{k}\left[Ric\,(Y,Z)X + Ric(Z,X)Y + Ric(X,Y)Z\right]. \tag{2.9}$$

Taking bi-covariant derivative of equation (2.9), with respect to 'U' and 'V', we get

$$\begin{aligned}(D_UD_VT)(X,Y,Z) = \frac{1}{k}\,&[(D_UD_VRic)\,(Y,Z)X + (D_UD_VRic)(Z,X)Y\\ &+(D_UD_VRic)(X,Y)Z].\end{aligned} \tag{2.10}$$

Now, let M_n be a generalized Ricci 2-recurrent Riemannian manifold, then using equation (1.9) in equation (2.10), we get

$$\begin{aligned}(D_UD_VT)(X,Y,Z) = \frac{1}{k}\,&[A(V)(D_URic)\,(Y,Z)X + B(U,V)Ric(Y,Z)X\\ &+A(V)(D_URic)\,(Z,X)Y + B(U,V)Ric(Z,X)Y\\ &+A(V)(D_URic)\,(X,Y)Z + B(U,V)Ric(X,Y)Z].\end{aligned} \tag{2.11}$$

On using equation (2.9) in equation (2.11), we get

$$(D_UD_VT)(X,Y,Z) = [A(V)(D_UT)\,(X,Y,Z) + B(U,V)T(X,Y,Z)]. \tag{2.12}$$

That is, M_n is 2-recurrent with respect to tensor $T(X,Y,Z)$.

Theorem (2.4). If an Einstein manifold M_n is generalized 2−recurrent with respect to the tensor $T(X,Y,Z)$, then

$$\begin{aligned}&\{(D_UD_VRic)(Y,Z) - A(V)(D_URic)(Y,Z) - B(U,V)Ric(Y,Z)\}X\\ &+\{(D_UD_VRic)(Z,X) - A(V)(D_URic)(Z,X) - B(U,V)Ric(Z,X)\}Y\\ &+\{(D_UD_VRic)(X,Y) - A(V)(D_URic)(X,Y) - B(U,V)Ric(X,Y)\}Z = 0.\end{aligned} \tag{2.13}$$

Proof. Let M_n be generalized 2-recurrent with respect to the tensor $T(X,Y,Z)$, then from equations (2.2) and (2.9), we have

$$A(V)(D_UT)(X,Y,Z) + B(U,V)T(X,Y,Z) = \frac{1}{k}[(D_UD_VRic)(Y,Z)X+$$

$$(D_U D_V Ric)(Z,X)Y + (D_U D_V Ric)(X,Y)Z]. \quad (2.14)$$

On using equation (1.9) in equation (2.14), we get the required result.

Theorem (2.5). An Einstein smooth Riemannian manifold M_n is generalized 2−recurrent with respect to the tensor $T(X,Y,Z)$, if and only if M_n is recurrent with respect to the tensor $S(X,Y,Z)$ for the same recurrence parameter.

Proof. From equations (2.8) and (1.4), we have

$$S(X,Y,Z) = kT(X,Y,Z). \quad (2.15)$$

Taking bi-covariant derivative of equation (2.15) with respect to 'U' and 'V', we get

$$(D_U D_V S)(X,Y,Z) = k(D_U D_V T)(X,Y,Z). \quad (2.16)$$

From equation (2.16), it is evident that, if M_n is 2-recurrent with respect to the tensor $S(X,Y,Z)$, then M_n is also 2-recurrent with respect to the tensor $T(X,Y,Z)$ and vice-versa.

We now prove the following :

Theorem (2.6). An $n-$dimensional smooth Riemannian manifold is 2-recurrent with respect to tensor $S(X,Y,Z)$, if it is Ricci 2-recurrent with the same recurrence parameter.

Proof. Taking bi-covariant derivative of equation (1.4) with respect to 'U' and 'V', we get

$$(D_U D_V S)(X,Y,Z) = (D_U D_V Ric)(Y,Z)X + (D_U D_V Ric)(Z,X)Y$$
$$+(D_U D_V Ric)(X,Y)Z. \quad (2.17)$$

Now, let M_n be Ricci 2-recurrent Riemannian manifold, then using equations (1.9) and (1.4) in equation (2.17), we get M_n as 2-recurrent Riemannian manifold with respect to the tensor $S(X,Y,Z)$.

Theorem (2.7). If a smooth Riemannian manifold M_n is 2-recurrent with respect to tensor $S(X,Y,Z)$, then

$$\{(D_U D_V Ric)(Y,Z) - A(V)(D_U Ric)(Y,Z) - B(U,V)Ric(Y,Z)\}X$$
$$+\{(D_U D_V Ric)(Z,X) - A(V)(D_U Ric)(Z,X) - B(U,V)Ric(Z,X)\}Y$$
$$+\{(D_U D_V Ric)(X,Y) - A(V)(D_U Ric)(X,Y) - B(U,V)Ric(X,Y)\}Z = 0.$$

Proof. Let M_n be 2-recurrent with respect to the tensor $S(X,Y,Z)$, then using equation (2.2) in equation (2.17), we have

$$A(V)(D_U S)(X,Y,Z) + B(U,V)S(X,Y,Z) = (D_U D_V Ric)(Y,Z)X$$

$$+(D_U D_V Ric)(Z,X)Y + (D_U D_V Ric)(X,Y)Z. \quad (2.18)$$

On using equation (1.4) in equation (2.18), we get the required results.

Corollary (2.1). An Einstein manifold M_n is 2-recurrent with respect to the tensor $T(X,Y,Z)$ if and only if M_n is 2-recurrent with respect to the tensor $F(X,Y,Z)$ for the same recurrence parameter.

3. Some Properties of Special Curvature Tensor $H(X,Y,Z)$ in Generalized 2-recurrent Smooth Riemannian Manifold

Theorem (3.1). In an $n-$dimensional smooth Riemannian manifold M_n, the special curvature tensor $H(X,Y,Z)$ has the following properties :

(i) If special curvature tensor $H(X,Y,Z)$ has cyclic property defined by equation (1.7), then it also has

$$\{(D_U H)(X,Y,Z) + (D_U H)(Y,Z,X) + (D_U H)(Z,Y,X)\} = 0,$$

and

$$(ii) \qquad (D_U D_X H)(Y,Z,W) + (D_U D_Y H)(Z,X,W) + (D_U D_Z H)(X,Y,W)$$

$$= (D_U D_X R)(Y,Z,W) + (D_U D_Y R)(Z,W,X) + (D_U D_Z R)(X,W,Y).$$

Proof.(i). Taking bi-covariant derivative of equation (1.7) with respect to 'U' and 'V', we get

$$(D_U D_V H)(X,Y,Z) + (D_U D_V H)(Y,Z,X) + (D_U D_V H)(Z,X,Y) = 0 \quad (3.1)$$

On using equations (2.1) and (1.7) in equation (3.1), we get the required result.

(ii) We have

$$H(Y,Z,W) = R(Y,Z,W) + R(Y,W,Z).$$

Taking bi-covariant derivative of the above equation with respect to 'X' and 'U', we get

$$(D_U D_X H)(Y,Z,W) = (D_U D_X R)(Y,Z,W) + (D_U D_X R)(Y,W,Z). \quad (3.2)$$

Taking cyclic permutation of equation (3.2) in X, Y, Z; adding the three equations and then using Bianchi's second identity, we get the required result.

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