J. T. S. ISSN : 0974-5428
Vol. 7 (2013), pp.1-8

On Extended Generalized Concircular $\phi-$Recurrent Lorentzian $\alpha-$Sasakian Manifold

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(Received: December 16, 2012)

Abstract

In this paper we have studied the extended generalized concircularly $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold and obtained some important results.

Keywords and Phrases : Lorentzian $\alpha-$Sasakian manifold, concircular $\phi-$recurrent, Scalar curvature.
2000 AMS Subject Classification : 53C05, 53C20, 53C25, 53D15.

1. Introduction

A transformation of an $n-$dimensional Riemannian manifold M, which transform every geodesic circle of M into a geodesic circle, is called a concircular transformation [8], [16]. A concircular transformation is always a conformal transformation. Here geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus, the geometry of concircular transformations, that is, the concircular geometry, is a generalization of inversive geometry in the sence that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor.

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. As a weaker version of local symmetry in 1977, Takahashi [15] introduced the notion of locally $\phi-$symmetric Sasakian manifold and obtained several interesting results. In 1979 Dubey [3] introduced the notion of generalized recurrent manifold and in 2007, Özgur [9], studied generalized recurrent Kenmotsu manifold. Generalizing this notion, Basari and Murathan [2], introduced the notion of generalized $\phi-$recurrency to Kenmotsu manifolds. Later in 2009, De, Yildiz and Yaliniz

[7], studied $\phi-$recurrent Kenmotsu manifolds, generalized $\phi-$recurrent Sasakian manifold and Lorentzian $\alpha-$Sasakian manifolds are studied in [10, 11]. Extending the notion of generalized $\phi-$recurrency, Shaikh and Hui [14], introduced the notion of extended generalized $\phi-$recurrency to $\beta-$Kenmotsu manifolds. The manifold M^n $(n > 2)$, is called generalized recurrent [3], if its curvature tensor R of type (1, 3) satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are nowhere vanishing unique $1-$forms defined by $A(.) = g(., \rho_1)$, $B(.) = g(., \rho_2)$ and G is a tensor of type (1, 3) given by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{1.2}$$

for all vector fields X, Y, $Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of all smooth vector fields on M and ∇ is the Levi-Civita connection.

Again M^n $(n > 2)$ is called generalized Ricci-recurrent manifold [5] if its Ricci tensor S of type (0, 2) satisfies the condition

$$\nabla S = A \otimes S + B \otimes g, \tag{1.3}$$

where A and B are nowhere vanishing unique $1-$forms.

This paper is organized as follows: Section 2 deals with a brief account of Lorentzian $\alpha-$Sasakian manifolds. In Section 3, we obtain the necessary and sufficient condition for a concircular manifold to be a generalized $\phi-$recurrent. Also it is shown that in a generalized concircularly $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold the vector field ρ_2 associated with the $1-$form B and the charecteristic vector field ξ are co-directional. Further, it is shown that the extended generalized concircularly $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold is a super generalized Ricci-recurrent manifold.

2. Preliminaries

A differentiable manifold M^n of dimension n is called a Lorentzian $\alpha-$ Sasakian manifold if it admits a $(1, 1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy:[11],

$$(a) \quad \eta(\xi) = -1, \qquad (b) \quad \phi(\xi) = 0, \qquad (c) \quad \eta(\phi X) = 0, \tag{2.1}$$

$$(a) \quad \phi^2 X = X + \eta(X)\xi, \quad (b) \quad g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi + \eta(Y)X), \tag{2.4}$$

for all $X, Y \in TM$.

Also a Lorentzian α-Sasakian manifold M^n satisfies,

$$\nabla_X \xi = \alpha\phi X, \tag{2.5}$$
$$(\nabla_X \eta)(Y) = \alpha g(X, \phi Y), \tag{2.6}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. Further, on a Lorentzian α-Sasakian manifold M^n, the following relations hold:[18],

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.7}$$
$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.8}$$
$$S(X,\xi) = (n-1)\alpha^2\eta(X), \tag{2.9}$$
$$Q\xi = (n-1)\alpha^2\xi, \tag{2.10}$$
$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\alpha^2\eta(X)\eta(Y), \tag{2.11}$$
$$(\nabla_W S)(Y,\xi) = \alpha((n-1)\alpha^2 g(W,\phi Y) - S(\phi W, Y)), \tag{2.12}$$
$$(\nabla_W R)(X,Y)\xi = \alpha[\alpha^2(g(\phi Y, W)X - g(\phi X, W)Y) - R(X,Y,\phi W)], \tag{2.13}$$

where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

Definition 2.1. A Lorentzian α-Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{2.14}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

Definition 2.2. A Lorentzian $\alpha-$Sasakian manifold is said to be generalized $\phi-$recurrent if its curvature tensor R satisfies the condition ([4])

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] \tag{2.15}$$

where A and B are two 1-forms and B is non-zero.

Definition 2.3. A Lorentzian $\alpha-$Sasakian manifold is said to be concircularly $\phi-$recurrent if there exists a nowhere vanishing unique $1-$form A such that

$$\phi^2((\nabla_W \overline{C})(X,Y)Z) = A(W)\overline{C}(X,Y)Z, \tag{2.16}$$

for all vector fields $X, Y, Z, W \in \chi(M)$.

3. Extended Generalized Concircular $\phi-$recurrent Lorentzian $\alpha-$ Sasakian Manifolds

A Lorentzian α-Sasakian manifold $M^n(\phi,\xi,\eta,g)$, is said to be an extended generalized concircular ϕ-recurrent if its concircular curvature tensor $\overline{C}$ satisfies the relation

$$\begin{aligned}\phi^2((\nabla_W\overline{C})(X,Y)Z) = A(W)\phi^2(\overline{C}(X,Y)Z)\\ +B(W)\phi^2(G(X,Y)Z),\end{aligned} \tag{3.1}$$

where A and B are non-vanishing 1-forms, ∇ denotes the operator of covariant differentiation with respect to the metric g i.e., ∇ is the Riemannian connection, and the Concircular curvature tensor $\overline{C}$ of type (1, 3) is given by

$$\overline{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}G(X,Y)Z, \tag{3.2}$$

where r is the scalar curvature of the manifold.

Let us consider an extended generalized concircularly $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold $M^n(\phi,\xi,\eta,g)$. Then by virtue of (2.2), it follows from (3.1) that

$$\begin{aligned}\phi^2((\nabla_W R)(X,Y)Z) - A(W)\phi^2(R(X,Y)Z) - B(W)\phi^2(G(X,Y)Z)\\ = \frac{dr(W) - rA(W)}{n(n-1)}[g(Y,Z)X - \eta(X)g(Y,Z)\xi\\ -g(X,Z)Y + \eta(Y)g(X,Z)\xi].\end{aligned} \tag{3.3}$$

This leads to the following:

Theorem 3.1. *An extended generalized concircularly ϕ-recurrent Lorentzian α-Sasakian manifold $M, n \geq 3$ is generalized ϕ-recurrent if and only if*

$$\begin{aligned}\frac{dr(W) - rA(W)}{n(n-1)}[g(Y,Z)X + \eta(X)g(Y,Z)\xi - g(X,Z)Y\\ -\eta(Y)g(X,Z)\xi] = 0.\end{aligned} \tag{3.4}$$

Now taking inner product of (3.4) with U, we have

$$\begin{aligned}\frac{dr(W) - rA(W)}{n(n-1)}[g(Y,Z)g(X,U) + \eta(X)g(Y,Z)\eta(U) - g(X,Z)g(Y,U)\\ -\eta(Y)g(X,Z)\eta(U)] = 0.\end{aligned}$$

contracting over X and U, we get

$$\{dr(W) - rA(W)\}[ng(Y,Z) - \eta(Y)\eta(Z)] = 0. \tag{3.5}$$

Again contracting (3.5) over Y and Z, we get

$$\{dr(W) - rA(W)\}[n^2 - 1] = 0. \tag{3.6}$$

which implies that

$$A(W) = \frac{1}{r} dr(W) \text{ for all } W \text{ and } r \neq 0 \tag{3.7}$$

$$\text{i.e.,} \quad \rho_1 = \frac{1}{r} \operatorname{grad} r, \text{ where } A(W) = g(W, \rho_1).$$

This leads to the following:

Theorem 3.2. If an extended generalized concircularly ϕ-recurrent Lorentzian α-Sasakian manifold M, $n \geq 3$ is generalized ϕ−recurrent, then the associated vector field corresponding to the 1−form A is given by $\rho_1 = \frac{1}{r}$ grad r, r being non-zero and non-constant scalar curvature of the manifold.

Now by virtue of (2.2), it follows from (3.3) that

$$\begin{aligned}(\nabla_W R)(X,Y)Z = &-\eta((\nabla_W R)(X,Y)Z)\xi + A(W)[R(X,Y)Z + \eta(R(X,Y)Z)\xi] \\ &+B(W)[G(X,Y)Z + \eta(G(X,Y)Z)\xi] + \frac{dr(W) - rA(W)}{n(n-1)} \\ &[g(Y,Z)X + \eta(X)g(Y,Z)\xi - g(X,Z)Y - \eta(Y)g(X,Z)\xi]. \end{aligned} \tag{3.8}$$

Taking inner product of (3.8) with U and then contracting over X and U, and using (2.4), (2.7) and (2.12), we get

$$\begin{aligned}(\nabla_W S)(Y,Z) = &\ A(W)S(Y,Z) + [nB(W) + \alpha^2 A(W)]g(Y,Z) \\ &+ \frac{dr(W)}{n(n-1)}[ng(Y,Z) - \eta(Y)\eta(Z)] + A(W)\Big[\Big(-\alpha^2 + \frac{r}{n(n-1)}\Big) \\ &\eta(Y)\eta(Z) - \frac{rn}{n(n-1)}g(Y,Z)\Big] - B(W)\eta(Y)\eta(Z). \end{aligned} \tag{3.9}$$

Again taking contraction over Y and Z in (3.9), we get

$$dr(W) = [r - \alpha^2 n(n-1)]A(W) - n(n^2-1)B(W). \tag{3.10}$$

From (3.10), we can state the following:

Theorem 3.3. In an extended generalized concircularly ϕ-recurrent Lorentzian α-Sasakian manifold M, $n \geq 3$, the associated 1-forms A and B are related by the relation (3.10)

Corollary 3.1. In an extended generalized concircularly ϕ-recurrent Lorentzian α-Sasakian manifold M, $n \geq 3$, with constant scalar curvature, the associated 1-forms A and B are related by

$$\{r - \alpha^2 n(n-1)\}A - n(n^2-1)B = 0$$

Now using (3.10) in (3.9), we get

$$\begin{aligned}(\nabla_W S)(Y,Z) = \; & A(W)S(Y,Z) + \{(-n^2)B(W) + \alpha^2(1-n)A(W)\}g(Y,Z) \\ & + nB(W)\eta(Y)\eta(Z). \qquad (3.11)\end{aligned}$$

From (3.11), it follows that the Ricci tensor S satisfies the condition

$$\nabla S = \alpha \otimes S + \beta \otimes g + \gamma \otimes \pi, \qquad (3.12)$$

where $\alpha(W) = A(W)$,

$$\beta(W) = -n^2 B(W) + \alpha^2(1-n)A(W), \quad \gamma(W) = nB(W)$$

and $\quad \pi = \eta \otimes \eta.$

From (3.12), we can state the following:

Theorem 3.4. An extended generalized concircular $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold M, $n \geq 3$, is super generalized Ricci-recurrent manifold.

Now contracting (3.11) over W and Z, we get

$$\frac{1}{2}dr(Y) = S(Y,\rho_1) - n^2 B(Y) + \alpha^2(1-n)A(Y) + n\eta(Y)B(\xi). \qquad (3.13)$$

By virtue of (3.10), the above relation takes the form

$$\begin{aligned}S(Y,\rho_1) = \; & [\frac{r - \alpha^2 n^2 + 3\alpha^2 n - 2\alpha^2}{2}]A(Y) \\ & -[\frac{n^3 - n + 2n^2}{2}]B(Y) - n\eta(Y)B(\xi). \qquad (3.14)\end{aligned}$$

From (3.14), we can state the following:

Theorem 3.5. In an extended generalized concircularly ϕ-recurrent Lorentzian α-Sasakian manifold M, $n \geq 3$, the Ricci tensor in the direction of ρ_1 is given by (3.14).

Now setting $Z = \xi$ in (3.11) and then using (2.9), we get

$$\alpha S(\phi W, Y) = \alpha^3(n-1)g(W,\phi Y) + n(n+1)B(W)\eta(Y). \qquad (3.15)$$

Replacing Y by ϕY in (3.15) and using (2.11) and (2.9), we have

$$\alpha S(W,Y) = \alpha^3(n-1)g(W,Y) \tag{3.16}$$

Replacing W by ϕW in (3.15) and then using (2.2), we get

$$\alpha S(W,Y) = \alpha^3(n-1)g(W,Y) + n(n+1)B(\phi W)\eta(Y). \tag{3.17}$$

From (3.16) and (3.17) we have

$$B(\phi W) = 0,$$

which implies that

$$B(W) = \eta(W)B(\xi).$$

This leads to the following:

Theorem 3.6. In an extended generalized concircularly ϕ-recurrent Lorentzian α-Sasakian manifold M, $n \geq 3$, the vector field ρ_2 associated with the 1-form B and the characteristic vector field ξ are codirectional.

References

1. Blair, D.E. : Contact manifolds in Riemannian geometry, Lecture Notes in Math. Springer 509 (1976).
2. Basari, A. and Murathan, C. : On generalized $\phi-$recurrent Kenmotsu manifolds, Fen Derg., 3(1)(2008), 91-97.
3. Dubey, R. S. D. : Generalized Recurrent Spaces, Indian Journal of Pure and Applied Mathematics, 10 (1979), 1508-1513.
4. De, U. C. and Guha, N. : On generalized recurrent manifolds, J. National Academy of Math. India, 9 (1991), 85-92.
5. De, U. C., Guha, N. and Kamilya, D. : On generalised Ricci-Recurrent Manifolds, Tensor, N. S., 56 (1995), 312-317.
6. De, U. C., Shaikh, A. A. and Biswas, S. : On $\phi-$recurrent Sasakian Manifolds, Novi Sad J. Math., 33 (2003), 13-48.
7. De, U. C., Yildiz, A. and Yaliniz, A. F. : On $\phi-$recurrent Kenmotsu Manifolds, Turk J. Math., 33 (2009), 17-25.
8. Kuhnel, W. : Conformal transformations between Einstein spaces, Conformal geometry (Bonn, 1985/1986), 105-146, Aspects Math., E12, Vieweg, Braunschweig, 1988.
9. Özgur, C. : On generalized recurrent Kenmotsu manifolds, World Appl. Sci. J., 2(1)(2007), 9-33.

10. Patil, D. A., Prakasha, D. G. and Bagewadi, C. S. : On Generalized $\phi-$recurrent Sasakian manifolds, Bulletin of Mathematical Analysis and Applications, 3 (2009), 1821-1291.
11. Prakasha, D. G. and Yildiz, A. : Generalized $\phi-$recurrent Lorentzian $\alpha-$Sasakian Manifold, Commun. Fac. Sci. Uni. Ank. Series A1., 59 (2010), 1303-5991.
12. Shaikh, A. A., Baishya, K. K. and Eyasmin, S. : On $\phi-$recurrent Generalised $(k, \mu)-$Contact Metric Manifolds, Lobachevski Journal of Mathematics, 27 (2007), 3-13.
13. Shaikh, Absos Ali and Ahmad, Helaluddin : On Generalized $\phi-$recurrent Sasakian Manifolds, Applied Mathematics, 2 (2011), 1317-1322.
14. Shaikh, Absos Ali and Hui, Shyamal Kumar : On Extended Generalized $\phi-$recurrent $\beta-$Kenmotsu Manifolds, Publications De Linstitut Mathematique, 89 (2011), 77-88.
15. Takahashi, T. : Sasakian $\phi-$symmetric spaces, Tohoku Math. J., 29 (1977), 91-113.
16. Yano, K. : Concircular geometry I. Concircular transformations, Proc. Imp. Acad. Tokyo, 16 (1940), 195-200.
17. Yildiz, A. and Murathan, C. : On Lorentzian $\alpha-$Sasakian manifolds, Kyungpook Math. J., 45 (2005), 95-103.
18. Yildiz, A., Turan, M. and Murathan, C. : A class of Lorentzian $\alpha-$Sasakian manifolds, Kyungpook Math. J., 49 (2009), 789-799.