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Douglas space of Second Kind of Finsler space with $(\alpha, \beta)-$Metric

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Abstract

The (α, β)-metric is a Finsler metric which is contstructed from a Riemannian metric α and a differential $1-$form β. In this paper we discussed the conditions under which the Finsler space with $(\alpha, \beta)-$metric $L = \beta^2/(\beta - \alpha)$ become a Douglas space of Second Kind.

Key Words: Douglas space, Finsler space, Douglas space of second kind, Berwald space, (α, β)-metric.
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1. Introduction

S. Basco and M. Matsumoto [4] have introduced the concept of Douglas space as a generalization of Berwald space. We also treat Landsberg space as a generalization of Berwald space. Recently, S. Basco and B. Szilagyi [5] introduced the notion of weakly Berwald space as another generalization of Berwald space. Berwald spaces with (α, β)-metric have been studied by several authors ([1], [11], [15], [14]). The present authors have studied on geometric properties of weakly Berwald space with some $(\alpha, \beta)-$metric [13]. The present paper gives a different definition of a Douglas space of the Finsler space with $(\alpha, \beta)-$metric, on the basis of Matsumoto's definition of Douglas space [13].

I.Y Lee [9] has worked on Douglas space of the second kind of Finsler space with Matsumoto metric. In the present paper we have established the result which gives the conditions under which Finsler space with $(\alpha, \beta)-$metric $L = \beta^2/(\beta - \alpha)$ become a Douglas space of second Kind.

2. Preliminaries

$F^n = (M^n, L(\alpha,\beta))$ is said to have an $(\alpha,\beta)-$metric $L(\alpha,\beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. The space $R^n = (M^n, \alpha)$ is called the Riemannian space associated with F^n([13], [12]). In R^n we have the Christoffel symbol $\gamma^i_{jk}(x)$ and the covariant differentiation $(;)$ with respect to γ^i_{jk}.

We use the following notations [9].

$$(a) \quad r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad r^i_j = a^{ih} r_{hj}, \quad r_j = b_i r^i_j,$$

$$(b) \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \quad s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j,$$

$$(c) \quad b^i = a^{ih} b_h, \quad b^2 = b^i b_i.$$

The Berwald connection $B\Gamma = \{G^i_{jk}, G^i_j\}$ of F^n plays one of the leading roles in the present paper. Denote by B^i_{jk} the difference tensor [11] of G^i_{jk} from γ^i_{jk}:

$$G^i_{jk}(x,y) = \gamma^i_{jk}(x) + B^i_{jk}(x,y).$$

Transvecting above equation by y^k, we get

$$G^i_j = \gamma^i_{0j} + B^i_j. \quad and \quad 2G^i = \gamma^i_{00} + 2B^i,$$

where $\gamma^i_{0j} = \gamma^i_{kj} y^k$ and $\gamma^i_{00} = \gamma^i_{jk} y^j y^k$ and then $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$. The following system of differential equations gives the geodesic of a Finsler space F^n in parameter t.

$$\ddot{x}^i \dot{x}^j - \ddot{x}^j \dot{x}^i + 2(G^i x^j - G^j x^i) = 0, \;\; y^i = \dot{x}^i.$$

The functions $G^i(x,y)$ are given by

$$2G^i(x,y) = \{j\;{}^i\;k\}\, y^j y^k,$$

where $\{j\;{}^i\;k\}$ are Christoffel symbols constructed from $g_{ij}(x,y)$ with respect to $\dot{x}^i$.

It is shown [4] that F^n is a Douglas space if and only if the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}(G_{ijk} y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

vanishes identically, where $G^h_{ijk} = \dot{\partial}_k G^h_{ij}$ is the hv-curvature tensor of the Berwald connection. F^n is said to be a Douglas space if

$$D^{ij} = G^i(x,y)y^j - G^j(x,y)y^i \tag{2.1}$$

are homogeneous polynomials in (y^i) of degree three.

Differentiating (2.1) with respect to y^h, y^k, y^p and y^q, we have $D^{ij}_{hkpq} = 0$, which are equivalent of $D^{im}_{hkpm} = (n+1)D^i_{hkp} = 0$. Thus if a Finsler space F^n satisfies the condition $D^{ij}_{hkpq} = 0$, which are equivalent to $D^{im}_{hkpm} = (n+1)D^i_{hkp} = 0$, we call it a Douglas space. Further differentiating (2.1) by y^m and contracting m and j in the obtained equation, we have $D^{im}_m = (n+1)G^i - G^m_m y^i$. Thus F^n is said to be a Douglas space of the second kind if and only if

$$D^{im}_m = (n+1)G^i - G^m_m y^i \tag{2.2}$$

are homogeneous polynomials in (y^i) of degree two. Again differentiating (2.2) with respect to y^h, y^j and y^k, we get $D^{im}_{hjkm} = (n+1)D^i_{hjk} = 0$. Hence we have

Definition 2.1.[9] If a Finsler space F^n satisfies the condition that $D^{im}_m = (n+1)G^i - G^m_m y^i$ be homogeneous polynomials in (y^i) of degree two, we call it a Douglas space of the second kind.

Definition 2.2.[9] A Finsler space with (α, β)-metric is said to be a Douglas space of the second kind if and only if

$$B^{im}_m = (n+1)B^i - B^m_m y^i$$

are homogeneous polynomials in (y^i) of degree two, where B^m_m is given by [8].

Furthermore differentiating the above with respect to y^h, y^j and y^k. we get another condition $B^{im}_{hjkm} = B^i_{hjk} = 0$.

The positive homogeneity of $L = L(\alpha, \beta)$ gives

$$\begin{aligned} &L_\alpha \alpha + L_\beta \beta = L, \quad L_{\alpha\alpha}\alpha + L_{\alpha\beta}\beta = 0, \\ &L_{\beta\alpha}\alpha + L_{\beta\beta}\beta = 0, \quad L_{\alpha\alpha\alpha}\alpha + L_{\alpha\alpha\beta}\beta = -L_{\alpha\alpha}, \\ &L_\alpha = \partial L/\partial\alpha, \quad L_\beta = \partial L/\partial\beta, \quad L_{\alpha\alpha} = \partial^2 L/\partial\alpha\partial\alpha, \\ &L_{\alpha\beta} = L_{\beta\alpha} = \partial^2 L/\partial\alpha\partial\beta, \quad L_{\alpha\alpha\alpha} = \partial^3 L/\partial\alpha\partial\alpha\partial\alpha. \end{aligned} \tag{2.3}$$

We use following lemma in next sections:

Lemma 2.1.[3] If $\alpha^2 \equiv 0(\operatorname{mod}\beta)$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfies $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.

3. Douglas space of the second kind with $(\alpha, \beta)-$metric

In this section, we study the condition that a Finsler space with an $(\alpha, \beta)-$metric be a Douglas space of the second kind.

Throughout the present paper, we say "homogeneous polynomial(s) in (y^i) of degree r" as $hp(r)$ for brevity. Consider the function $G^i(x,y)$ of F^n with an $(\alpha,\beta)-$metric. According to ([11], [12]) $G^i(x,y)$ can be written in the form

$$\begin{aligned} 2G^i &= \gamma^i_{00} + 2B^i, \\ B^i &= (E/\alpha)y^i + (\alpha L_\beta/L_\alpha)s^i_0 - (\alpha L_{\alpha\alpha}/L_\alpha)C^*\{(y^i/\alpha) - (\alpha/\beta)b^i\}, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} E &= (\beta L_\beta/L)C^*, \\ C^* &= \{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)\} \,/\, \left\{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})\right\}, \\ \gamma^2 &= b^2\alpha^2 - \beta^2. \end{aligned}$$

Since $\gamma^i_{00} = \gamma^i_{jk}(x)y^iy^j$ is $hp(2)$, by means of (2.1) and (3.1) we have as follows[13]: A Finsler space F^n with an (α,β)-metric is a Douglas space if and only if $B^{ij} = B^iy^j - B^jy^i$ are $hp(3)$. (2.1) gives

$$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^iy^j - b^jy^i). \tag{3.2}$$

Differentiating above equation by y^m and contracting m and j in the obtained equation, we get

$$\begin{aligned} B^{im}_m &= \dot{\partial}_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)(s^i_0y^m - s^m_0y^i) + \frac{\alpha L_\beta}{L_\alpha}\dot{\partial}_m(s^i_0y^m - s^m_0y^i) \\ &+ \dot{\partial}_m\left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)C^*(b^iy^m - b^my^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}(\dot{\partial}_mC^*)(b^iy^m - b^my^i) \\ &+ \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}C^*\dot{\partial}_m(b^iy^m - b^my^i). \end{aligned} \tag{3.3}$$

Using (2.2) and the homogeneity of (y^i), we obtain

$$\dot{\partial}_m\left(\frac{\alpha L_\beta}{L_\alpha}\right)(s^i_0y^m - s^m_0y^i) = \left(\frac{\alpha L_\beta}{L_\alpha}\right)s^i_0 - \frac{\alpha^2 L L_{\alpha\alpha}s_0}{(\beta L_\alpha)^2}y^i, \tag{3.4}$$

$$\frac{\alpha L_\beta}{L_\alpha}\dot{\partial}_m(s^i_0y^m - s^m_0y^i) = \frac{n\alpha L_\beta}{L_\alpha}s^i_0, \tag{3.5}$$

$$\dot{\partial}_m\left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)(b^iy^m - b^my^i)C^* = \frac{\gamma^2\{\alpha L_\alpha L_{\alpha\alpha\alpha} + (2L_\alpha - \alpha L_{\alpha\alpha})L_{\alpha\alpha}\}C^*}{(\beta L_\alpha)^2}y^i, \tag{3.6}$$

$$(\dot{\partial}_mC^*)y^m = 2C^*, \tag{3.7}$$

$$(3.8)\qquad (\dot\partial_m C^*)b^m = \frac{1}{2\alpha\beta\Omega^2}[\Omega\{\beta(\gamma^2+2\beta^2)M + 2\alpha^2\beta^2 L_\alpha r_0 - \alpha\beta\gamma^2 L_{\alpha\alpha} r_{00} - 2\alpha(\beta^3 L_\beta + \alpha^2\gamma^2 L_{\alpha\alpha})s_0\} - \alpha^2\beta M\{2b^2\beta^2 L_\alpha - \gamma^4 L_{\alpha\alpha\alpha} - b^2\alpha\gamma^2 L_{\alpha\alpha}\}],$$

$$(3.9)\qquad \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* \dot\partial_m (b^i y^m - b^m y^i) = \frac{(n-1)\alpha^2 L_{\alpha\alpha} C^*}{\beta L_\alpha} b^i,$$

where

$$(3.10)\qquad \begin{aligned} M &= (r_{00}L_\alpha - 2\alpha s_0 L_\beta),\\ \Omega &= (\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \quad \text{provided that } \Omega \neq 0,\\ Y_i &= a_{ir}y^r,\ s_{00}=0,\ b^r s_r = 0,\ a^{ij}s_{ij}=0. \end{aligned}$$

Plugging (3.4), (3.5), (3.6), (3.7), (3.8) and (3.9) in (3.3), we get

$$(3.11)\qquad \begin{aligned} B^{im}_m &= \frac{(n+1)\alpha L_\beta}{L_\alpha} s^i_0 + \frac{\alpha(n+1)\alpha^2\Omega L_{\alpha\alpha} b^i + \beta\gamma^2 A y^i}{2\Omega^2} r_{00}\\ &- \frac{\alpha^2(n+1)\alpha^2\Omega L_\beta L_{\alpha\alpha} b^i + B y^i}{L_\alpha \Omega^2} s_0 - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\Omega} r_0, \end{aligned}$$

where

$$(3.12)\qquad \begin{aligned} A &= \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha(L_{\alpha\alpha})^2,\\ B &= \alpha\beta\gamma^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\{(3\gamma^2-\beta^2)L_\alpha - 4\alpha\gamma^2 L_{\alpha\alpha}\}L_\beta L_{\alpha\alpha} + \Omega L L_{\alpha\alpha}. \end{aligned}$$

Finally from above equations, we establish

Theorem 3.1.[9] The necessary and sufficient condition for a Finsler space F^n with an (α,β)-metric to be a Douglas space of second kind is that B^{im}_m are homogeneous polynomials in (y^m) of degree two, where B^{im}_m is given by (3.11) and (3.12), provided that $\Omega \neq 0$.

4. Douglas space of the second kind with $L = \frac{\beta^2}{\beta-\alpha}$

In this section we consider the Finsler space with (α,β)-metric $L = \frac{\beta^2}{\beta-\alpha}$, for this metric we get the partial derivatives

$$(4.1)\qquad \begin{aligned} &L_\alpha = \frac{\beta^2}{(\beta-\alpha)^2}, \quad L_\beta = \frac{\beta(\beta-2\alpha)}{(\beta-\alpha)^2},\\ &L_{\alpha\alpha} = \frac{2\beta^2}{(\beta-\alpha)^3}, \quad L_{\alpha\alpha\alpha} = \frac{6\beta^2}{(\beta-\alpha)^4},\\ &\Omega = \frac{\beta^2}{(\beta-\alpha)^3}\{\beta^2(\beta-3\alpha) + 2b^2\alpha^3\}. \end{aligned}$$

Plugging (4.1) in (3.12), we get

$$A = \frac{6\beta^4(\beta - 2\alpha)}{(\beta - \alpha)^6},$$

$$B = \frac{2\beta^6}{(\beta - \alpha)^8}\{14b^2\alpha^4 - 12b^2\alpha^3\beta + (3b^2 - 15)\alpha^2\beta^2 + 13\alpha\beta^3 - 3\beta^4\}. \tag{4.2}$$

Again inserting (4.1) and (4.2) in (3.11), we get

$$\begin{aligned}
&[4b^4\alpha^6\beta - 12b^2\alpha^4\beta^3 + 4b^2\alpha^3\beta^4 + 9\alpha^2\beta^5 - 6\alpha\beta^6 + \beta^7]B^{im}_m \\
&- [-8(n+1)b^4\alpha^8 + 4(n+1)b^4\alpha^7\beta + 24b^2(n+1)\alpha^6\beta^2 \\
&- 20b^2(n+1)\alpha^5\beta^3 + 4b^2(n+1)\alpha^4\beta^4 + 21(n+1)\alpha^3\beta^5 \\
&- 8(n+1)\alpha^2\beta^6 + (n+1)\alpha\beta^7]s^i_0 - \{[2b^2(n+1)\alpha^6\beta - 3(n+1)\alpha^4\beta^3 \\
&+ (n+1)\alpha^3\beta^4]b^i - [6\alpha^2\beta^2\gamma^2 - 3\alpha\beta^3\gamma^2]y^i\}r_{00} \\
&+ \{[-8(n+1)b^2\alpha^8 + 4b^2(n+1)\alpha^7\beta \\
&+ 12(n+1)\alpha^6\beta^2 - 10(n+1)\alpha^5\beta^3 + 2(n+1)\alpha^4\beta^4]b^i \\
&+ [28b^2\alpha^6\beta - 24b^2\alpha^5\beta^2 + 2(3b^2 - 15)\alpha^4\beta^3 \\
&+ 26\alpha^3\beta^4 - 6\alpha^2\beta^5]y^i\}s_0 + [4b^2\alpha^6\beta - 6\alpha^4\beta^3 + 2\alpha^3\beta^4]y^i r_0 = 0.
\end{aligned} \tag{4.3}$$

Assume that F^n be a Douglas space of the second kind, that is B^{im}_m be $hp(2)$. Since α is irrational in (y^i), the above equation is divided into two parts as follows:

$$\begin{aligned}
&[4b^4\alpha^6\beta - 12b^2\alpha^4\beta^3 + 9\alpha^2\beta^5 + \beta^7]B^{im}_m - [-8(n+1)b^4\alpha^8 \\
&+ 24b^2(n+1)\alpha^6\beta^2 + 4b^2(n+1)\alpha^4\beta^4 - 8(n+1)\alpha^2\beta^6]s^i_0 \\
&- \{[2b^2(n+1)\alpha^6\beta - 3(n+1)\alpha^4\beta^3]b^i + [-6\alpha^2\beta^2\gamma^2]y^i\}r_{00} \\
&+ \{[-8(n+1)b^2\alpha^8 + 12(n+1)\alpha^6\beta^2 + 2(n+1)\alpha^4\beta^4]b^i \\
&+ [28b^2\alpha^6\beta + 2(3b^2 - 15)\alpha^4\beta^3 - 6\alpha^2\beta^5]y^i\}s_0 \\
&+ [4b^2\alpha^6\beta - 6\alpha^4\beta^3]y^i r_0 = 0.
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&[4b^2\alpha^2\beta^3 - 6\beta^5]B^{im}_m - [4(n+1)b^4\alpha^6 - 20b^2(n+1)\alpha^4\beta^2 \\
&+ 21(n+1)\alpha^2\beta^4 + (n+1)\beta^6]s^i_0 - \{(n+1)\alpha^2\beta^3 + 3\beta^2\gamma^2 y^i\}r_{00} \\
&+ \{[4b^2(n+1)\alpha^6 - 10(n+1)\alpha^4\beta^2]b^i \\
&+ [-24b^2\alpha^4\beta + 26\alpha^2\beta^3]y^i\}s_0 + 2\alpha^2\beta^3 y^i r_0 = 0.
\end{aligned} \tag{4.5}$$

Since only the term $\beta^7 B^{im}_m$ of (4.4) seemingly does not contain α^2, we must have $hp(7)V^i_7$ such that $\beta^7 B^{im}_m = \alpha^2 V^i_7$.

First we deal with the case $\alpha^2 \not\equiv 0(\mathrm{mod}\,\beta)$, that is, $n > 2$. Then there exist a function $g^i(x)$ such that $V_7^i = g^i(x)\beta^7$, then

$$B^{im}_m = \alpha^2 g^i(x). \tag{4.6}$$

The term not containing α^2 in (4.5) is $-6\beta^5 B^{im}_m - (n+1)\beta^6 s^i_0 - 3\beta^2\gamma^2 y^i r_{00}$, hence there must exist $hp(5)U^i_5$ such that

$$-\beta^2[6\beta^3 B^{im}_m + (n+1)\beta^4 s^i_0 + 3\gamma^2 y^i r_{00}] = \alpha^2 U^i_5.$$

Above shows that there exist $hp(3)U^i_3$ such that $U^i_5 = \beta^2 U^i_3$, which implies

$$-[6\beta^3 B^{im}_m + (n+1)\beta^4 s^i_0 + 3\gamma^2 y^i r_{00}] = \alpha^2 U^i_3.$$

Substituting (4.6) in above equation we get

$$-[(n+1)\beta^4 s^i_0 + 3\gamma^2 y^i r_{00}] = \alpha^2[U^i_3 + 6\beta^3 g^i(x)].$$

Since $\alpha^2 \not\equiv 0(\mathrm{mod}\,\beta)$, $U^i_3 + 6\beta^3 g^i(x)$ must vanish, and hence

$$-3\gamma^2 y^i r_{00} = (n+1)\beta^4 s^i_0. \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.4), we get

$$\begin{aligned}
&[4b^4\alpha^6\beta - 12b^2\alpha^4\beta^3 + 9\alpha^2\beta^5 + \beta^7]g^i(x) - [-8(n+1)b^4\alpha^6 \\
&+ 24b^2(n+1)\alpha^4\beta^2 + 4b^2(n+1)\alpha^2\beta^4 - 8(n+1)\beta^6]s^i_0 \\
&- \{[2b^2(n+1)\alpha^4\beta - 3(n+1)\alpha^2\beta^3]b^i\}r_{00} + 2(n+1)\beta^6 y^i s^i_0 \\
&+ \{[-8(n+1)b^2\alpha^6 + 12(n+1)\alpha^4\beta^2 + 2(n+1)\alpha^2\beta^4]b^i \\
&+ [28b^2\alpha^4\beta + 2(3b^2-15)\alpha^2\beta^3 - 6\beta^5]y^i\}s_0 \\
&+ [4b^2\alpha^4\beta - 6\alpha^2\beta^3]y^i r_0 = 0.
\end{aligned} \tag{4.8}$$

Only the term $\beta^7 g^i(x) + 10(n+1)\beta^6 s^i_0 - 6\beta^5 y^i s_0$ of (4.8) seemingly does not contain α^2, and hence we must have $hp(5)V^i_5$ such that $\beta^5[\beta^2 g^i(x) + 10(n+1)\beta s^i_0 - 6y^i s_0] = \alpha^2 V^i_5$. If $V^i_5 = h^i(x)\beta^5$, then $\beta^2 g^i(x) + 10(n+1)\beta s^i_0 - 6y^i s_0 = \alpha^2 h^i(x)$. Transvecting by b_i gives $\beta^2 g^i(x)b_i + 10(n+1)\beta s_0 - 6s_0 = \alpha^2 h_b$, where $b_i h^i = h_b$. Thus we get $h_b = 0$, which gives

$$s_0 = \frac{-\beta^2 g_b}{2[5(n+1)\beta - 3]}. \tag{4.9}$$

Substituting (4.6), (4.7) and (4.9) in (4.5), we get

$$\begin{aligned}(4.10)\quad & [4b^2\alpha^2\beta^3 - 6\beta^5]\alpha^2 g^i(x) - [4(n+1)b^4\alpha^6 - 20b^2(n+1)\alpha^4\beta^2 \\ & + 21(n+1)\alpha^2\beta^4 + (n+1)\beta^6]s_0^i - \{(n+1)\alpha^2\beta^3\}r_{00} + (n+1)\beta^6 s_0^i \\ & + \{[4b^2(n+1)\alpha^6 - 10(n+1)\alpha^4\beta^2]b^i \\ & + [-24b^2\alpha^4\beta + 26\alpha^2\beta^3]y^i\}\frac{-\beta^2 g_b}{2[5(n+1)\beta - 3]} + 2\alpha^2\beta^3 y^i r_0 = 0.\end{aligned}$$

Since only the term $-2(n+1)\beta^6 s_0^i$ seemingly does not contain α^2, there exist $hp(5)V_5^i$ such that $-2(n+1)\beta^6 s_0^i = \alpha^2 V_5^i$, then $-2(n+1)\beta s_0^i = \alpha^2 h^i(x)$. Suppose that $s_0 = 0$, then because $\alpha^2 \not\equiv 0(\mathrm{mod}\,\beta)$, $s_0^i = 0$. Thus

Theorem 4.1. A Finsler space with the metric $\beta^2/(\beta-\alpha)$ is a Douglas space of second kind, if and only if $\alpha^2 \not\equiv 0(\mathrm{mod}\,\beta)$: (4.6) and (4.7) are satisfied, and $s_0^i = 0$.

Now consider $\alpha^2 \equiv 0(\mathrm{mod}\,\beta)$, Lemma 2.1 shows that $n = 2$, $b^2 = 0$ and $\alpha^2 = \beta\delta$, $\delta = d_i(x)y^i$. From these conditions (4.4) and (4.5) are rewritten in the form

$$\begin{aligned}(4.11)\quad & [\beta^2 + 9\beta\delta]B_m^{im} + (n+1)\beta^2\delta s_0^i + [3(n+1)\delta^2 b^i + 6\delta y^i]r_{00} \\ & +\{[2(n+1)\beta\delta^2 + 12(n+1)\delta^3]b^i - [30\delta^2 + 6\beta\delta]y^i\}s_0 - 6\delta^2 y^i r_0 = 0\end{aligned}$$

$$\begin{aligned}(4.12)\quad & 6\beta B_m^{im} + [(n+1)\beta^2 + 9(n+1)\beta\delta + 12(n+1)\beta\delta - 8(n+1)\delta^2]s_0^i \\ & +[(n+1)\delta b^i - 3y^i]r_{00} - [-10(n+1)\delta^2 b^i + 26\delta y^i]s_0 - 2\delta y^i r_0 = 0\end{aligned}$$

The term $\beta^2 B_m^{im}$ of (4.11) seemingly does not contain δ, hence there must exist $hp(1)U^i$ such that

$$(4.13)\qquad B_m^{im} = \delta U^i,$$

Paying attention to the terms which do not contain δ in (4.12), we have

$$6\beta B_m^{im} + (n+1)\beta^2 s_0^i - 3y^i r_{00} = \delta V_2^i,$$

Substituting (4.13) in the above equation

$$(4.14)\qquad 3y^i r_{00} = (n+1)\beta^2 s_0^i - \delta[V_2^i - 6\beta U^i].$$

Substituting (4.13) and (4.14) in (4.11) we get

$$\begin{aligned}& [\beta^2 + 9\beta\delta]U^i + (n+1)\beta^2 s_0^i + [3(n+1)\delta b^i + 6y^i]r_{00} \\ & +\{[2(n+1)\beta\delta + 12(n+1)\delta^2]b^i - [30\delta + 6\beta]y^i\}s_0 - 6\delta y^i r_0 = 0\end{aligned}$$

Since the terms $\beta^2 U^i + (n+1)\beta^2 s^i_0 + 6y^i r_{00} - 9\beta y^i s_0$ seemingly do not contain δ, there must exist $hp(2) U^i_2$ such that $\beta^2 U^i + 3(n+1)\beta^2 s^i_0 - 9\beta y^i s_0 = \delta U^i_2$, which implies

$$3(n+1)\beta^2 s^i_0 = \delta U^i_2 + 9\beta y^i s_0 - \beta^2 U^i. \tag{4.15}$$

Substituting (4.13) and (4.14) and (4.15) in (4.12), we get

$$\begin{aligned}
&6\beta\delta U^i + [(n+1)\beta^2 + 9(n+1)\beta\delta + 12(n+1)\beta\delta - 8(n+1)\delta^2] \\
&\times \frac{\delta U^i_2 + 9\beta y^i s_0 - \beta^2 U^i}{3(n+1)\beta^2} \\
&+[(n+1)\delta b^i - 3y^i]\frac{(n+1)\beta^2 s^i_0 - \delta[V^i_2 - 6\beta U^i]}{3y^i} \\
&-[-10(n+1)\delta^2 b^i + 26\delta y^i]s_0 - 2\delta y^i r_0 = 0
\end{aligned}$$

Term not containing δ in the above equation is $\dfrac{\beta(9y^i s_0 - \beta U^i)}{3} s_0 - (n+1)\beta^2 s^i_0$, hence there exist $hp(2) V^i_2$ such that $\beta(9y^i s_0 - \beta U^i)s_0 - \beta(9y^i s_0 - \beta U^i)s_0 = \delta V^i_2$ which implies $V^i_2 = 0$. Thus

Theorem 4.2. A Finsler space with the metric $\beta^2/(\beta - \alpha)$ is a Douglas space of second kind, if and only if $\alpha^2 \equiv 0 (\mathrm{mod}\, \beta)$: (4.13), (4.14) and (4.15) are satisfied.

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