J. T. S. ISSN : 0974-5428
Special Vol. 8 (2014), pp.103-111

# $\tau$-Curvature On Kenmotsu Manifold

**Venkatesha* and Vishnuvardhana. S. V.**

Department of Mathematics
Kuvempu University,Shankaraghatta - 577 451,
Shimoga, Karnataka, India
e-mail: vensmath@gmail.com, svvishnuvardhana@gmail.com
(**corresponding author*)
(Received: November 11, 2013)

## Abstract

In the present paper we have obtained the necessary and sufficient condition for a extended generalized $\tau$-$\phi$-recurrent Kenmotsu manifold to be a generalized ricci-recurrent manifold. Furthermore, we have studied $\tau$-$\phi$-symmetric Kenmotsu manifold, $\tau$-$\xi$-flat Kenmotsu manifold and a Kenmotsu manifold satisfying $\tau(X, Y) \cdot R = 0$.

**Keywords and Phrases :** $\tau$-curvature tensor, $\tau$-$\phi$-symmetric Kenmotsu manifold, extended generalized $\tau$-$\phi$-recurrent Kenmotsu manifold, $\eta$-Einstein manifold, $\tau$-$\xi$-flat Kenmotsu manifold.
**2000 AMS Subject Classification :** 53C05, 53C20, 53C50.

## Introduction

As is well known, symmetric spaces plays an important role in differential geometry. The work on local symmetric Riemannian manifolds began by Cartan [2]. This property of a Riemannian manifold has been weakened by many authors [ [22], [18], [3], [7], [19], [16], [17] ] in several directions such as recurrent manifolds, semi-symmetric manifolds, pseudo-symmetric manifolds, weakly symmetric manifolds. Further, $\phi$-recurrent, generalized $\phi$-recurrent, extended generalized $\phi$-recurrent manifolds were introduced and studied by many geometers.

In 1979 Dubey [8] introduced the notion of generalized recurrent manifold and then such a manifold was studied by De and Guha [6]. The manifold $M$, $n > 2$, is called generalized recurrent [8] if its curvature tensor R of type (1,3) satisfies the condition $\nabla R = A \otimes R + B \otimes G$, where $G$ is a tensor of type (1,

3) given by $G(X,Y)Z = g(Y,Z)X - g(X,Z)Y$, $A$ and $B$ are nowhere vanishing unique 1-forms, defined by $A(\cdot) = g(\cdot,\rho_1)$ and $B(\cdot) = g(\cdot,\rho_2)$, respectively for all vector fields $X$, $Y$, $Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of all smooth vector fields on $M$ and $\nabla$ is the Levi-Civita connection.

In 1952 Patterson [15] introduced Ricci-recurrent manifold. According to Patterson [15] manifold $(M,g)$ of dimension $n$, is called a Ricci-recurrent if $(\nabla_X S)(Y,Z) = A(X)S(Y,Z)$, for some 1-form $A$. Ricci-recurrent manifold has been studied by many authors. An extended version of ricci-recurrent manifold is the generalized Ricci-recurrent manifold. A non-flat Riemannian manifold called generalized Ricci-recurrent [5] if its Ricci tensor $S$ of type (0, 2) satisfies the condition $\nabla S = A \otimes S + B \otimes g$, where $A$ and $B$ are two non zero 1-forms. In particular, if $B = 0$, then the manifold reduces to Ricci-recurrent manifold.

Our work is structured as follows: The first section is a very brief review of Kenmotsu manifold and a $\tau$-curvature tensor. The next section is devoted to the study of $\tau$-$\phi$-symmetric kenmotsu manifold for two cases $\tau = 0$ and $\tau \neq 0$. For $\tau = 0$, the Kenmotsu manifold is $\tau$-$\phi$-symmetric provided either $r$ is a constant or $a_7 = 0$, this is obviously true if the manifold is $W_i$ flat$(i = 0, \cdots 9)$ or $W_j^*$ flat $(i = 0, 1)$ or conhormonically flat or projectively flat or $M$-projectively flat. For $\tau \neq 0$ case it is shown that any two conditions of (i) $M^{2n+1}$ is $\phi$-$\tau$-symmetric, (ii) $M^{2n+1}$ is $\phi$-symmetric, (iii) either $a_7 = 0$ or $r$ is constant, are true then the remaining statement holds. In Section 3, we have proved that the extended generalized $\tau$-$\phi$-recurrent Kenmotsu manifold is generalized Ricci-recurrent manifold with $a_0 + 2na_1 + a_2 + a_3 \neq 0$ and viceversa. Finally, it is shown that $\tau$-$\xi$-flat Kenmotsu manifold is a $\eta$-Einstein manifold provided $a_4 \neq 0$ and has a scalar curvature $r$ if $a_4 = 0$ and $a_7 \neq 0$. Moreover we have proved that Kenmotsu manifold satisfying $\tau(X,Y) \cdot R = 0$ is a $\eta$-Einstein manifold.

## 1. Preliminaries

Kenmotsu manifold has been introduced and studied by K. Kenmotsu in 1972 [9]. They set up one of the three classes of almost contact metric manifolds whose automorphism group attains the maximum dimension [20]. For such a manifold, the sectional curvature of a plane sections containing $\xi$ is a constant, say $c$. It has been studied as homogeneous normal contact Riemannian manifolds if $c > 0$. Global Riemannian products of a line or a circle with a Kahler manifold of constant holomorphic sectional curvature with $c = 0$, and a warped product space $R \times_f C_n$, if $c < 0$. Kenmotsu [9] characterized the differential geometric properties of manifold for $c < 0$ and the structure so obtained is now known as Kenmotsu structure. A Kenmotsu structure is not Sasakian. Manifolds of $c > 0$ are characterized by some tensor equations, it has a Sasakian structure. Manifolds with $c = 0$ are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu obtained some tensorial equations to characterize manifolds of $c < 0$.

An almost contact metric manifold is a differentiable manifold $M^{2n+1}$ endowed with a structure $(\phi, \xi, \eta, g)$ given by a tensor field $\phi$ of type (1,1), a vector field $\xi$, a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta\, o\, \xi, \qquad \eta(\xi) = 1, \tag{1.1}$$

and a Riemannian metric $g$ such that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields $X$ and $Y$. The fundamental 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, \phi Y)$for any vector fields $X$ and $Y$. It is well known that contact metric manifolds are almost contact metric manifolds such that $\Phi = d\eta$.

Thus a manifold $M^{2n+1}$ equipped with this structure is called an almost contact manifold and is denoted by $(M^{2n+1}, \phi, \xi, \eta)$. If $g$ is a Riemannian metric on an almost contact manifold $M^{2n+1}$ such that,

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{1.2}$$

$$\nabla_X \xi = (X - \eta(X)\xi), \tag{1.3}$$

holds, then $(M^{2n+1}, \phi, \xi, \eta)$ is called Kenmotsu manifold. Here $\nabla$ denotes the operator of covariant differentiation with respect to $g$.

In a Kenmotsu manifold $M^{2n+1}$, the following relations holds;

$$\eta(R(X, Y)Z) = [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{1.4}$$

$$\text{(a) } R(\xi, X)Y = [\eta(Y)X - g(X, Y)\xi], \text{ (b) } R(X, Y)\xi = [\eta(X)Y - \eta(Y)X], \tag{1.5}$$

$$\text{(a) } S(X, Y) = -2ng(X, Y), \text{ (b) } S(X, \xi) = -2n\eta(X), \text{ (c) } QX = -2nX, \tag{1.6}$$

$$\text{(a) } S(\xi, \xi) = -2n, \text{ (b) } Q\xi = -2n\xi, \tag{1.7}$$

$$(\nabla_W R)(X, Y)\xi = g(W, X)Y - g(W, Y)X - R(X, Y)W, \tag{1.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y). \tag{1.9}$$

In a $(2n + 1)$-dimensional Riemannian manifold $M^{2n+1}$, the $\tau$-curvature tensor [10] is given by

$$\begin{aligned} \tau(X, Y)Z = {} & a_0 R(X, Y)Z + a_1 S(Y, Z)X + a_2 S(X, Z)Y + a_3 S(X, Y)Z \\ & + a_4 g(Y, Z)QX + a_5 g(X, Z)QY + a_6 g(X, Y)QZ \\ & + a_7 r\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned} \tag{1.10}$$

where $R, S, Q$ and $r$ are the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature respectively. In particular, $\tau$-curvature tensor is reduces to quasi-conformal curvature tensor C*, conformal curvature tensor C, conharmonic curvature tensor L, concircular curvature tensor V, pseudo-projective curvature tensor P*, projective curvature tensor P, $M$-projective curvature tensor, $W_i$-curvature tensors $(i = 0, \cdots, 9)$ and $W_j^*$-curvature tensors $(j = 0, 1)$, by assigning particular values to $a_i's$ $(i = 0, 1, \cdots, 7)$ in the equation (1.10).

## 2. $\tau$-$\phi$-symmetric Kenmotsu manifold

**Definition 2.1.** A Kenmotsu manifold $M^{2n+1}$ is said to be $\phi$-symmetric [12] if the condition $\phi^2((\nabla_W R)(X, Y)Z) = 0$ holds, for all vector fields $X, Y, Z \in \chi(M)$.

Taking $\tau = 0$ in (1.10) and using (1.6), we get

$$-a_0 R(X, Y)Z = -2n(a_1 + a_4)g(Y, Z)X - 2n(a_2 + a_5)g(X, Z)Y$$
$$-2n(a_3 + a_6)g(X, Y)Z + a_7 r\{g(Y, Z)X - g(X, Z)Y\}.$$

On covariant differentiation of the above equation with respect to $W$, and assuming that all vector fields $X$, $Y$, $Z$, $W$ are orthogonal to $\xi$, one can get

$$-a_0((\nabla_W R)(X, Y)Z) = -a_7 dr(W)g(Y, Z)X + a_7 dr(W)g(X, Z)Y.$$

i.e.,

$$((\nabla_W R)(X, Y)Z) = \frac{a_7}{a_0} dr(W)\{g(X, Z)Y - g(Y, Z)X\}.$$

Applying $\phi^2$ on both sides of the above equation and using (1.1), we get

$$\phi^2((\nabla_W R)(X, Y)Z) = \frac{a_7}{a_0} dr(W)\{g(X, Z)Y - g(Y, Z)X\}.$$

Therefore we can state;

**Theorem 2.1.** A $\tau$-flat Kenmotsu manifold is $\phi$-symmetric provided either $r$ is constant or $a_7 = 0$.

From Theorem (2.1), we have the following corollary;

**Corollary 2.1.** A Kenmotsu manifold is $\phi$-symmetric if either $r$ is constant or manifold is $W_i$ flat $(i = 0, \cdots 9)$ or $W_j^*$ flat $(i = 0, 1)$ or conhormonically flat or projectively flat or $M$-projectively flat.

And if $\tau \neq 0$, then we arrive at

$$\phi^2((\nabla_W \tau)(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z) + \frac{a_7}{a_0} dr(W)\{g(Y, Z)X - g(X, Z)Y\}.$$

And so we can state;

**Theorem 2.2.** If in a Kenmotsu manifold $M^{2n+1}$ any two of the following statements hold then the remaining statement holds
(a) $M^{2n+1}$ is $\phi$-$\tau$-symmetric.
(b) $M^{2n+1}$ is $\phi$-symmetric.
(c) Either $a_7 = 0$ or $r$ is constant.

## 3. Extended generalized $\tau$-$\phi$- recurrent Kenmotsu manifold

**Definition 3.1.** A Kenmotsu manifold is said to be a extended generalized $\tau$-$\phi$-recurrent manifold if there exists non-zero 1-forms $A$ and $B$ such that

$$\phi^2((\nabla_W \tau)(X,Y)Z) = A(W)\phi^2(\tau(X,Y)Z) + B(W)\phi^2(G(X,Y)Z), \qquad (3.1)$$

for arbitrary vector fields $X, Y, Z, W$. If $X, Y, Z, W$ are orthogonal to $\xi$, then the manifold is called locally $\phi$-recurrent manifold. If the 1 -form $A$ vanishes, then the manifold reduces to $\phi$-symmetric manifold.

**Theorem 3.1.** Extended generalized $\tau$-$\phi$- recurrent Kenmotsu manifold $M^{2n+1}$, with $a_0 + 2na_1 + a_2 + a_3 \neq 0$ is generalized Ricci-recurrent if and only if the following relation holds:

$$\frac{\{B(W) - A(W)(a_0 - 2n(a_2 + a_3 + a_5 + a_6) - a_7 r) + a_7 dr(W)\}}{(a_0 + 2na_1 + a_2 + a_3)}\eta(Y)\eta(Z)$$
$$- \frac{\{a_2[S(W,Z) + 2ng(W,Z)]\eta(Y) - a_3[S(W,Y) + 2ng(W,Y)]\eta(Z)\}}{(a_0 + 2na_1 + a_2 + a_3)}$$
$$+ \frac{a_5\eta(Z)\eta((\nabla_W Q)Y) + a_6\eta(Y)\eta((\nabla_W Q)Z)}{(a_0 + 2na_1 + a_2 + a_3)} = 0. \qquad (3.2)$$

**Proof.** By taking an account of Definition 3.2 and using (1.1), we obtain

$$-(\nabla_W \tau)(X,Y)Z + \eta((\nabla_W \tau)(X,Y)Z)\xi = A(W)[-\tau(X,Y)Z + \eta(\tau(X,Y)Z)\xi]$$
$$+B(W)[-G(X,Y)Z + \eta(G(X,Y)Z)\xi].$$

i.e.,

$$g((-\nabla_W \tau)(X,Y)Z, U) + \eta((\nabla_W \tau)(X,Y)Z)\eta(U)$$
$$= A(W)[g(-\tau(X,Y)Z, U) + \eta(\tau(X,Y)Z)\eta(U)]$$
$$+B(W)[-g(G(X,Y)Z, U) + \eta(G(X,Y)Z)\eta(U)].$$

Let $\{e_i : i = 1, 2, 3, \cdots, 2n+1\}$ be an othonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in the above and taking summation over $i$, $1 \leq i \leq 2n+1$ and using (1.10), we have

$$-g((\nabla_W \tau)(e_i, Y)Z, e_i) + \eta((\nabla_W \tau)(e_i, Y)Z)\eta(e_i)$$
$$= A(W)[-g(\tau(e_i, Y)Z, e_i) + \eta(\tau(e_i, Y)Z)\eta(e_i)] \quad (3.3)$$
$$+B(W)[-g(G(e_i, Y)Z, e_i) + \eta(G(e_i, Y)Z)\eta(e_i)].$$

Using the equations (1.3), (1.6), (1.8) and (1.10) and the symmetric property of Ricci-tensor and the relation $g((\nabla_W R)(X,Y)Z,U) = -g((\nabla_W R)(X,Y)U,Z)$, we have

$$(\nabla_W S)(Y,Z) =$$
$$\frac{\{B(W) - A(W)(a_0 - 2n(a_2 + a_3 + a_5 + a_6) - a_7 r) + a_7 dr(W)\}}{(a_0 + 2na_1 + a_2 + a_3)}\eta(Y)\eta(Z)$$
$$+\frac{\{(2n-1)B(W) - (a_4 + (2n-1)a_7)dr(W) - A(W)((2n-1)a_7 r}{(a_0 + 2na_1 + a_2 + a_3)}$$
$$\frac{+(r+2n)a_4 + a_0)\}}{}g(Y,Z) + [A(W) + \frac{A(W)(a_5 + a_6)}{a_0 + 2na_1 + a_2 + a_3}]S(Y,Z) \quad (3.4)$$
$$+\frac{a_5\eta(Z)\eta((\nabla_W Q)Y) + a_6\eta(Y)\eta((\nabla_W Q)Z)}{(a_0 + 2na_1 + a_2 + a_3)}$$
$$-\frac{\{a_2[S(W,Z) + 2ng(W,Z)]\eta(Y) - a_3[S(W,Y) + 2ng(W,Y)]\eta(Z)\}}{(a_0 + 2na_1 + a_2 + a_3)}.$$

The above relation will reduces to $\nabla S = A^* S + B^* g$ only when the relation (3.2) holds,

where $A^* = [A(W) + \dfrac{A(W)(a_5 + a_6)}{a_0 + 2na_1 + a_2 + a_3}]$

and $B^* = \dfrac{\{(2n-1)B(W) - (a_4 + (2n-1)a_7)dr(W) - A(W)((2n-1)a_7 r}{(a_0 + 2na_1 + a_2 + a_3)}$

$$\frac{+(r+2n)a_4 + a_0)\}}{}.$$

## 4. $\tau$-$\xi$-flat Kenmotsu manifold

Putting $Y = Z = \xi$ in (1.10) and taking inner product with U, we obtain

$$\tau(X,\xi,\xi,U) = a_0 R(X,\xi,\xi,U) + a_1 S(\xi,\xi)g(X,U) + a_2 S(X,\xi)g(\xi,U)$$
$$+a_3 S(X,\xi)g(\xi,U) + a_4 g(\xi,\xi)g(QX,U) + a_5 g(X,\xi)g(Q\xi,U)$$
$$+a_6 g(X,\xi)g(Q\xi,U) + a_7 r(g(\xi,\xi)g(X,U) - g(X,\xi)g(\xi,U)). \quad (4.1)$$

By virtue of (1.5), (1.6) and the condition of $\tau$-$\xi$-flat in (4.1), we get

$$0 = a_0\eta(U)\eta(X) - a_0 g(X,U) + (a_7 r - 2na_1)g(X,U)$$
$$-2n(a_2 + a_3 + a_5 + a_6)\eta(X)\eta(U) + a_4 S(X,U) - a_7 r\eta(X)\eta(U). \quad (4.2)$$

Simplifying (4.2), we get

$$S(X,U) = \frac{(a_0 + 2na_1 - a_7 r)}{a_4} g(X,U) + \frac{(a_7 r + 2n(a_2 + a_3 + a_5 + a_6) - a_0)}{a_4}\eta(X)\eta(U).$$

Thus we have the following theorem;

**Theorem 4.1.** A $\tau$-$\xi$-flat Kenmotsu manifold is $\eta$-Einstein provided $a_4 \neq 0$.

Now suppose $a_4 = 0$ and $a_7 \neq 0$ then from (4.2), we have

$$0 = (a_0 + 2na_1 - a_7 r)g(X,U) + (a_7 r + 2n(a_2 + a_3 + a_5 + a_6) - a_0)\eta(X)\eta(U).$$

Contracting the above, we get

$$r = \frac{(a_2 + a_3 + a_5 + a_6) + a_0 + (2n+1)a_1}{a_7}. \quad (4.3)$$

Therefore,

**Theorem 4.2.** In a $\tau$-$\xi$-flat Kenmotsu manifold with $a_4 = 0$ and $a_7 \neq 0$, the scalar curvature $r$ is given by (4.3).

Next if $a_4 = 0$ and $a_7 = 0$ then (4.2) gives

$$0 = -(2na_1 + a_0)g(X,U) + (a_0 - 2n(a_2 + a_3 + a_5 + a_6))\eta(X)\eta(U),$$

on contraction, we have

$$2n(a_2 + a_3 + a_5 + a_6 + a_0 + (2n+1)a_1) = 0. \quad (4.4)$$

Thus we can state;

**Theorem 4.3.** In $\tau$-$\xi$-flat Kenmotsu manifold with $a_4 = 0$ and $a_7 = 0$, $2n(a_2 + a_3 + a_5 + a_6 + a_0 + (2n+1)a_1) = 0$.

**Theorem 4.4.** A Kenmotsu manifold $M^{2n+1}$, satisfying $\tau(X,Y)\cdot R = 0$, is an Einstein manifold provided $a_1 \neq 0$.

**Proof.** Suppose $\tau(X,Y)\cdot R = 0$. Then we have

$$\tau(X,Y)R(U,V)W - R(\tau(X,Y)U,V)W - R(U,\tau(X,Y)V)W$$
$$-R(U,V)\tau(X,Y)W = 0. \quad (4.5)$$

Putting $X = \xi$ in (4.5) and then taking inner product with $\xi$, we obtain

$$g(\tau(\xi,Y)R(U,V)W,\xi) - g(R(\tau(\xi,Y)U,V)W,\xi)$$
$$-g(R(U,\tau(\xi,Y)V)W,\xi) - g(R(U,V)\tau(\xi,Y)W,\xi) = 0. \tag{4.6}$$

By virtue of (1.4), (1.5), (1.6) and (4.6), we get

$$2n(a_2+a_4+a_5)g(U,Y)\eta(V)\eta(W) - 2n(a_2+a_4+a_5)g(V,Y)\eta(U)\eta(W)$$
$$-a_1S(Y,U)\eta(V)\eta(W) - a_1S(Y,V)\eta(U)\eta(W)$$
$$+4n(a_3+a_6)g(U,W)\eta(Y)\eta(V) - 4n(a_3+a_6)g(V,W)\eta(Y)\eta(U). \tag{4.7}$$

Contracting (4.7), we have

$$S(Y,V) = \frac{2n(a_2+a_4+a_5)}{a_1}g(Y,V)$$
$$+\frac{2n\{a_1-(a_2+a_4+a_5)-4n(a_3+a_6)}{a_1}\eta(Y)\eta(V).$$

## References


[1] **Blair, D. E. :** Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, 509 Springer-Verlag, Berlin, 1976.

[2] **Cartan, E. :** Surune classe remarquable despaces de Riema, Soc. Math., France, 54(1926), 214-264.

[3] **Chaki, M. C. :** Some theorems on recurrent and Ricci-recurrent spaces, Rend Semin Math Della Univ Di Padova., 26(1956), 168-176.

[4] **Chaubey, S. K. and Ojha, R. H. :** On the m-projective curvature tensor of a Kenmotsu manifold, Diff. Geom. Dyna. Syst., 12 (2010), 52-60.

[5] **De, U. C., Guha, N. and Kamaliya, D. :** On generalized Ricci-recurrent manifolds, Tensor N. S., 56 (1995), 312-317.

[6] **De, U. C. and Guha, N. :** On generalized recurrent manifolds, J. Nat. Acad. Math. India., 9 (1991), 85-92.

[7] **Deszcz, R. :** On pseudosymmetric spaces, Acta Math., Hungarica, 53 (1992), 185-190.

[8] **Dubey, R. S. :** Generalized recurrent spaces, Indian J. Pure. Appl. Math., 10 (1979), 1508-1513.

[9] **Kenmotsu, K. :** A class of almost contact Riemannian manifolds, Tôhoku Math. J., 24(1972), 93-103.

[10] **Tripathi, M. M. and Gupta, Punam :** T-curvature tensor on a semi- Riemannian manifold, J. Adv. Math. Stud., 4 (2011), 117-129.

[11] **Tripathi, M. M. and Gupta, Punam :** On T-curvature tensor in K-contact and Sasakian manifolds, Int. Electron. J. Geom., 4 (1)(2011), 32-47.

[12] **Nagaraja, H. G. and Somashekhara, G. :** $\tau$-curvature tensor in $(k,\mu)$-contact manifolds, Mathematica Aeterna, 2 (2012), no. 6, 523-532.

[13] **Oubina, J. A. :** New classes of contact metric structures, Publ. Math. Debrecen., 32 (1985), 187-193.

[14] **Prakasha, D. G. :** On extended generalized $\phi$-recurrent Sasakian manifolds, Journal of the Egyptian Mathematical Society, 21 (2013), 25-31.

[15] **Patterson, E. M. :** Some theorems on Ricci-recurrent spaces, J. London Math. Soc., 27 (1952), 287-295.

[16] **Selberg, A. :** Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to dirichlet series, Indian Math. Soc., 20 (1956), 47-87.

[17] **Shaikh, A. A. and Hui, S. K. :** On extended generalized $\phi$-recurrent $\beta$- Kenmotsu manifolds, Publ. De LInst. Math., 89 (2011), 77-88.

[18] **Szabo, Z. I. :** Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version, J. Differential Geom., 17 (1982), 531-582.

[19] **Tamassy, L. and Binh, T. Q. :** On weakly symmetric and weakly projective symmetric Riemannian manifold, Colloquia Math. Soc., 50 (1989), 663-667.

[20] **Tanno, S. :** The automorphism groups of almost contact Riemannian manifolds, Tohoku Math. J., 21 (1969), 21-38.

[21] **Venkatesha and Bagewadi, C. S. :** On concircular $\phi$-recurrent LP-Sasakian manifolds, Differ. Geom. Dyn. Syst., 10 (2008), 312-319.

[22] **Walker, A. A. :** On Ruses spaces of recurrent curvature, Proc. London Math. Soc., 52 (1950), 36-64.