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On Concircularly $\phi-$Recurrent Lorentzian $\alpha-$Sasakian Manifold with Semi-Symmetric Non-Metric Connection

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Abstract

The present study deals with concirurlar recurrent Lorentzian $\alpha-$Sasakian manifolds with semi symmetric non-metric connection and discuss some interesting results.

1. Introduction

The notion of local symmetry of a Riemannian manifold has been weaked by many author in several ways to a different extent. As a weaker version of local symmetry, Takahashi [12] introduced the notion of local $\phi-$symmetry on a Saskian manifold. Generalizing the notion of $\phi-$symmetry. De, U. C. [6] introduced the notion of $\phi-$recurrent Sasakian manifold.

Fridmann and Schouten introduced the idea of semi symmetric linear connection on a differentiable manifold. Hayden [1] introduced the idea of metric connection with torsion on Riemannian manifold. further, Yano [14], Golab [8] defined and studied semi symmetric and quarter symmetric connection with affine connection, further many author like De, U. C. [5], Sharfuddin and Hussain [2], Rastogi, Mishra and Pandey, Babewadi and many other studies the various properties of semi-symmetric connection.

In this paper we studied Concirurlar $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection and proved that a $\phi-$ recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection is $\eta-$Einstein manifold. Further we have shown that in $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection the characteristics vector ξ and vector field ρ associated to the 1-form A are codirectional. Finally, we proved that concircular $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection is $\eta-$Einstein manifold.

2. Preliminaries

A differentiable manifold M of dimension n is called a Lorentzian $\alpha-$Sasakian manifold if it admits a tensor field ϕ of type (1, 1), a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy

$$\phi^2 = 1 + \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \tag{2.5}$$

$$(D_X\phi)Y = \alpha g(X, Y)\xi - \alpha\eta(Y)X, \tag{2.6}$$

for all $X, Y \in Tm$ [2, 3, 13].

Also a Lorentzian $\alpha-$Sasakian manifold m satisfies

$$(D_X\xi)Y = \alpha\phi X, \tag{2.7}$$

$$(D_X\eta)Y = -\alpha g(\phi X, Y) \tag{2.8}$$

where D denotes the operator of covariant differentiation with respect to Lorentzian matric g. Also on a Lorentzian $\alpha-$Sasakian manifold the following relation hold [2, 3, 13]

$$R(X, Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{2.9}$$

$$R(\xi, X)Y = \alpha^2(g(X, Y)\xi - \eta(Y)X), \tag{2.10}$$

$$R(\xi, X)\xi = \alpha^2(\eta(X)\xi + X), \tag{2.11}$$

$$S(X, \xi) = (n - 1)\alpha^2\eta(X), \tag{2.12}$$

$$\eta(R(X, Y)Z) = \alpha^2(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)), \tag{2.13}$$

$$g(R(\xi, X)Y, \xi) = -\alpha^2[g(X, Y) + \eta(X)\eta(Y)]. \tag{2.14}$$

For any vector field X, Y, Z where S is the Ricci curvature and Q is the Ricci operator given by

$$S(X, Y) = g(\phi X, Y).$$

A Lorentzian $\alpha-$Sasakian manifold is said to be $\eta-$Einstein manifold if its Ricci tensor S takes the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for arbitray vector X, Y where a and b are function on M. If $b = 0$, the $\eta-$Einstein manifold becomes Einstein manifold. [3,9] have proved that if Lorentzian $\alpha-$Sasakian manifold M is $\eta-$Einstein manifold then $a+b = -\alpha^2(n-1)$.

Definition (2.1). A Lorentzian $\alpha-$Sasakian manifold is said to be locally $\phi-$symmetric if

$$\phi^2((D_W R)(X,Y)Z) = 0. \tag{2.15}$$

Definition (2.2). A Lorentzian $\alpha-$Sasakian manifold is said to be recurrent if there exists a non zero 1-form A such that

$$\phi^2((D_W R)(X,Y)Z) = A(W)R(X,Y)Z, \tag{2.16}$$

where $A(W)$ is defined by $A(W) = g(W,\rho)$ and ρ is a vector field associated with the 1-form.

Definition (2.3). A Lorentzian $\alpha-$Sasakian manifold is said to be concircularly recurrent if there exists a non zero 1-form A such that

$$\phi^2((D_W C)(X,Y)Z) = A(W)C(X,Y)Z, \tag{2.17}$$

where C is the concircular curvature tensor given by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2.18}$$

where R is the Riemannian curvature tensor and r is the scalar curvature.

A linear connection D in an $n-$dimensional differentiable manifold is said to be semi-symmetric connection if its torsion tensor is of the form [1,5,8]

$$T(X,Y) = D_X Y - D_Y X - [X,Y] = \eta(Y)X - \eta(X)Y, \tag{2.19}$$

for all X, Y on TM. A semi-symmetric connection D is said to be semi-symmetric non-metric connection if it further satisfies $D_X g \neq 0$. [8]

3. Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection

A semi-symmetric non-metric connection $\overline{D}$ in Lorentzian $\alpha-$Sasakian manifold can be defined by

$$\overline{D}_X Y = D_X Y + \eta(Y)X. \tag{3.1}$$

Also, we have

$$(\overline{D}_X g)(Y,Z) = -\eta(Y)g(X,Z) - \eta(Z)g(Y,X). \tag{3.2}$$

A connection given by (3.1) with (3.2) is called semi-symmetric non-metric connection in Lorentzian $\alpha-$Sasakian manifold.

A relation between curvature tensor M of the manifold with semi-symmetric non-metric conection $\overline{D}$ and Levi-Civita connection D is given by

$$\overline{R}(X,Y)Z = R(X,Y)Z - \alpha g(\phi X, Z)Y - \alpha g(\phi Y, Z)X, \tag{3.3}$$

where $\overline{R}$ and R are the Riemannian curvature of the connection $\overline{D}$ and D respectively.

From (3.3), we have

$$\overline{S}(Y,Z) = S(Y,Z) + \alpha(n-1)g(\phi Y, Z), \tag{3.4}$$

where $\overline{S}$ and S are the Ricci tensor of the connection $\overline{D}$ and D respectively.

Contracting (3.4), we get

$$\overline{r} = r, \tag{3.5}$$

where $\overline{r}$ and r are the scalar curvatures of the connection $\overline{D}$ and D respectively.

4. $\phi-$Recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection

A analogous to the definition (2.2)we define a Lorentzian $\alpha-$Sasakian manifold is said to be $\phi-$recurrent with respect to semi-symmetric non-metric connection if its curvature tensor $\overline{R}$ satifies the following condition

$$\phi^2((D_W\overline{R})(X,Y)Z) = A(W)\overline{R}(X,Y)Z. \tag{4.1}$$

Using (2.1) in (4.1), we get

$$(\overline{D}_W\overline{R})(X,Y)Z + \eta((\overline{D}_W\overline{R})(X,Y)Z)\xi = A(W)\overline{R}(X,Y)Z. \tag{4.2}$$

From which it follows that

$$g((\overline{D}_W\overline{R})(X,Y)Z,U) + \eta((\overline{D}_W\overline{R})(X,Y)Z)g(\xi,U) = A(W)g(\overline{R}(X,Y)Z,U). \tag{4.3}$$

Let $\{e_i\}$, $i = 1,2,3,\ldots n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = \{e_i\}$ in (4.3) and taking summation over i, $1 \leq i \leq n$, we get

$$(\overline{D}_W\overline{S})(Y,Z) + \eta((\overline{D}_W\overline{R})(e_i,Y)Z)\eta(e_i) = A(W)\overline{S}(Y,Z). \tag{4.4}$$

Putting $Z = \xi$ in (4.4), the second term of (4.4) takes the form $g((\overline{D}_W\overline{R})(e_i,Y)\xi, \xi)$ which on simplification gives $g((\overline{D}_W\overline{R})(e_i,Y)\xi,\xi) = 0$.

Then from (4.4) we obtain

$$(\overline{D}_W\overline{S})(Y,\xi) = A(W)\overline{S}(Y,\xi). \tag{4.5}$$

Now, we know that

$$(\overline{D}_W\overline{S})(Y,\xi) = \overline{D}_W\overline{S}(Y,\xi) - \overline{S}(\overline{D}_WY,\xi) - \overline{S}(Y,\overline{D}_W\xi). \tag{4.6}$$

Using (2.7), (2.8), (2.12), (3.4) in (4.6), we get

$$\begin{aligned}(\overline{D}_W\overline{S})(Y,\xi) =& \alpha S(Y,\phi W) + S(Y,W) - \alpha(\alpha+1)(n-1)g(Y,\phi W)\\ &- \alpha^2(n-1)g(Y,W) + \alpha^2(n-1)g(\phi Y,\phi W).\end{aligned} \tag{4.7}$$

In view of (4.5) and (4.7), we get

$$\begin{aligned}&\alpha S(Y,\phi W) + S(Y,W) - \alpha(\alpha+1)(n-1)g(Y,\phi W) - \alpha^2(n-1)g(Y,W)\\ &+ \alpha^2(n-1)g(\phi Y,\phi W) = \alpha^2(n-1)A(W)\eta(Y).\end{aligned}$$

Replacing $Y = \phi Y$ in above equation, we get

$$\begin{aligned}&\alpha S(\phi Y,\phi W) + S(\phi Y,W) - \alpha(\alpha+1)(n-1)g(\phi Y,\phi W)\\ &- \alpha^2(n-1)g(\phi Y,W) + \alpha^2(n-1)g(Y,\phi W) = 0.\end{aligned} \tag{4.8}$$

Interchanging Y and W in (4.8), we get

$$\begin{aligned}&\alpha S(\phi W,\phi Y) + S(\phi W,Y) - \alpha(\alpha+1)(n-1)g(\phi W,\phi Y)\\ &- \alpha^2(n-1)g(\phi W,Y) + \alpha^2(n-1)g(W,\phi Y) = 0.\end{aligned} \tag{4.9}$$

Adding (4.8) and (4.9) and simplifying, we get

$$S(\phi Y,\phi W) = (\alpha^2+1)(n-1)g(\phi Y,\phi W).$$

Using (2.3) and (2.15), we get

$$S(Y,W) = (\alpha^2+1)(n-1)g(Y,W) + (n-1)\eta(Y)\eta(W).$$

This leads to the following theorem:

Theorem (4.1). A $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection in $\eta-$Einestien manifold.

Again (4.2), we have

$$(\overline{D}_W\overline{R})(X,Y)Z = -\eta((\overline{D}_W\overline{R})(X,Y)Z)\xi + A(W)\overline{R}(X,Y)Z. \tag{4.10}$$

From (2.13), (3.3) and using Bainchi identity, we get

$$A(W)\eta(\overline{R}(X,Y)Z) + A(X)\eta(\overline{R}(Y,W)Z) + A(Y)\eta(\overline{R}(W,X)Z) = 0. \tag{4.11}$$

From (2.13), (3.3) in (4.11), we get

$$A(W)\alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)\alpha^2[g(Z,W)\eta(Y)$$

$$- g(Y,Z)\eta(W)] + A(W)\alpha^2[g(X,W)\eta(Z) - g(Z,W)\eta(X)] + \alpha[g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y) + g(\phi W,Z)\eta(Y) - g(\phi Y,Z)\eta(W) + g(\phi X,Z)\eta(W) - g(\phi W,Z)\eta(X)] = 0. \tag{4.12}$$

Putting $Y = Z = e_i$ in (4.12) and taking summation over i, $1 \leq i \leq n$, we get

$$A(W)\eta(X) = A(X)\eta(W) \tag{4.13}$$

for all vector fields W. Replacing X by ξ in (4.13), we get

$$A(W) = -\eta(\rho)\eta(W), \tag{4.14}$$

for any vector field W, where $A(\xi) = g(\xi,\rho) = \eta(\rho)$, ρ being vector field associated to the 1-form A that is $g(X,\rho) = A(X)$.

From (4.13) and (4.14), we state the following:

Theorem (4.2). In a $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection the characteristic vector ξ and vector field ρ associated to the 1-form A are codirectional and 1-form A is given by (4.14).

5. Concircular $\phi-$Recurrent Lorentzian $\alpha-$Sasakian manifold with Semi-symmetric Non-metric Connection

A analogous to the definition (2.3), we define a Lorentzian $\alpha-$Sasakian manifold is said to be concircular $\phi-$recurrent with respect to semi-symmetric non-metric connection if

$$\phi^2((D_W\overline{C})(X,Y)Z) = A(W)\overline{C}(X,Y)Z, \tag{5.1}$$

where $\overline{C}$ is the concircular curvature tensor with respect to semi-symmetric non-metric connection given by

$$\overline{C}(X,Y)Z = \overline{R}(X,Y)Z - \frac{\overline{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.2}$$

Using (2.1) in (5.1), we have

$$(\overline{D}_W\overline{C})(X,Y)Z + \eta((\overline{D}_W\overline{C})(X,Y)Z\xi = A(W)\overline{C}(X,Y)Z. \tag{5.3}$$

From which it follows that

$$g((\overline{D}_W\overline{C})(X,Y)Z,U) + \eta((\overline{D}_W\overline{C})(X,Y)Z)g(\xi,U) = A(W)g(\overline{C}(X,Y)Z,U). \tag{5.4}$$

Let $\{e_i\}$, $i = 1,2,3,\dots n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = \{e_i\}$ in (4.3) and taking summation over i, $1 \leq i \leq n$, we get

$$
\begin{aligned}
(\overline{D}_W\overline{S})(Y,Z) =& \frac{\overline{dr}(w)}{n}g(Y,Z) - \frac{\overline{dr}(w)}{n(n-1)}[g(Y,Z)-\eta(Y)\eta(Z)] \\
&+ A(W)\Big[\overline{S}(Y,Z) - \frac{\overline{r}}{n}g(Y,Z)\Big].
\end{aligned}
$$

Putting $Z=\xi$ and using (2.2) and (2.4) in above, we have

$$
(\overline{D}_W\overline{S})(Y,\xi) = \frac{\overline{dr}(w)}{n}\eta(Y) - \frac{\overline{dr}(w)}{n(n-1)}2\eta(Y) + A(W)\Big[\overline{S}(Y,\xi) - \frac{\overline{r}}{n}\eta(Y)\Big]. \quad (5.5)
$$

Now, we know that

$$
(\overline{D}_W\overline{S})(Y,\xi) = \overline{D}_W\overline{S}(Y,\xi) - \overline{S}(\overline{D}_WY,\xi) - \overline{S}(Y,\overline{D}_W\xi). \quad (5.6)
$$

Using (2.7) (2.8),(2.12), (3.4) in (5.6), we get

$$
\begin{aligned}
(\overline{D}_W\overline{S})(Y,\xi) =& \alpha S(Y,\phi W) + S(Y,W) - \alpha(\alpha+1)(n-1)g(Y,\phi W) \\
&- \alpha^2(n-1)g(Y,W) + \alpha^2(n-1)g(\phi Y,\phi W).
\end{aligned} \quad (5.7)
$$

In view of (5.5) and (5.7), we get

$$
\begin{aligned}
&\alpha S(Y,\phi W) + S(Y,W) - \alpha(\alpha+1)(n-1)g(Y,\phi W) - \alpha^2(n-1)g(Y,W) \\
&+ \alpha^2(n-1)g(\phi Y,\phi W) = \frac{\overline{dr}(w)}{n}\eta(Y) - \frac{\overline{dr}(w)}{n(n-1)}2\eta(Y) \\
&+ A(W)\Big[\overline{S}(Y,\xi) - \frac{\overline{r}}{n}\eta(Y)\Big].
\end{aligned}
$$

Replacing $Y=\phi Y$ in above equation, we get

$$
\begin{aligned}
&\alpha S(\phi Y,\phi W) + S(\phi Y,W) - \alpha(\alpha+1)(n-1)g(\phi Y,\phi W) \\
&- \alpha^2(n-1)g(\phi Y,W) + \alpha^2(n-1)g(Y,\phi W) = 0.
\end{aligned} \quad (5.8)
$$

Interchanging Y and W in (5.8), we get

$$
\begin{aligned}
&\alpha S(\phi W,\phi Y) + S(\phi W,Y) - \alpha(\alpha+1)(n-1)g(\phi W,\phi Y) \\
&- \alpha^2(n-1)g(\phi W,Y) + \alpha^2(n-1)g(W,\phi Y) = 0.
\end{aligned} \quad (5.9)
$$

Adding (5.8) and (5.9) and simplifying we get

$$
S(\phi Y,\phi W) = (\alpha^2+1)(n-1)g(\phi Y,\phi W).
$$

Using (2.3) and (2.15), we get

$$
S(Y,W) = (\alpha^2+1)(n-1)g(Y,W) + (n-1)\eta(Y)\eta(W).
$$

This leads to the following theorem:

Theorem (5.1). A concircular $\phi-$recurrent Lorentzian $\alpha-$Sasakian manifold with semi-symmetric non-metric connection is $\eta-$Einestien manifold.

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