J. T. S. ISSN : 0974-5428
Vol. 11 (2017), pp. 97-105

Petrov Type D Spacetimes and Lanczos Potential

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(Received: November 14, 2017)

Abstract

Using the Newman-Penrose formalism, the Lanczos potential for Petrov type D spacetimes has been found. It is seen that the Lanczos scalars can be expressed in terms of the spin-coefficients. The non-uniqueness character of the Lanczos potential has been established and a possible justification to the name "the Lanczos potential" is given.

1. Introduction

C. Lanczos generated the gravitational field through the equation (c.f.,[27], [32])

$$C_{hijk} = L_{[hi][j;k]} + L_{[jk][h;i]} - {}^*L^*{}_{[hi][j;k]} - {}^*L^*{}_{[jk][h;i]} \tag{1}$$

where dual operation is applied to each pair of skew-symmetric brackets as indicated. The double dual is defined as ${}^*A^*{}_{hijk} = \frac{1}{4}\eta_{hilm}\eta_{jkno}A^{lmno}$ and L_{ijk} is a rank three tensor field which satisfies the following properties

$$L_{ijk} \ = \ -L_{jik} \tag{2}$$

$$L_i{}^t{}_t \ = \ 0 \text{ (or, } g^{kl}L_{kil} = 0) \tag{3}$$

$$L_{ijk} \ + \ L_{jki} \ + \ L_{kij} \ = \ 0 \text{ (or, } {}^*L_i{}^t{}_t = 0) \tag{4}$$

$$L_{ij}{}^k{}_{;k} \ = \ 0 \text{ [or, } (L^k_{ij} \ + \ L_{ij}{}^{jk})_{;k}] \ = \ 0 \tag{5}$$

Eqn (1) can also be expressed as ([14])

$$C_{hijk} = L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + L_{(hk)}g_{ij} + L_{(ij)}g_{hk} - L_{(hj)}g_{ik} - L_{(ik)}g_{hj}$$
$$+\frac{2}{3}L^{pq}{}_{p;q}(g_{hj}g_{ik} - g_{hk}g_{ij}) \tag{6}$$

where

$$L_{hk} = L^t{}_{hk;t} - L^t{}_{ht;k} \tag{7}$$

and round bracket denotes symmetrization.

Using eqns (3), (5) and (7), eqn (6) can also be written as

$$C_{hijk} = L_{hij;k} - L_{hik;j} + L_{jkh;i} - L_{jki;h} + \frac{1}{2}(L^p{}_{hk;p} + L^p{}_{kh;p})g_{ij} + \frac{1}{2}(L^p{}_{ij;p} + L^p{}_{ji;p})g_{hk}$$

$$-\frac{1}{2}(L^p{}_{hj;p} + L^p{}_{jh;p})g_{ik} - \frac{1}{2}(L^p{}_{ik;p} + L^p{}_{ki;p})g_{hj} \tag{8}$$

Eqn (1)/(6)/(8) is known as *Weyl-Lanczos equation* and the tensor field L_{ijk} is now commonly known as *Lanczos potential* or *Lanczos spin tensor.* The role of Lanczos potential L_{ijk} with respect to Weyl tensor C_{hijk} (gravitational field) is same as that of the vector potential for the electromagnetic field tensor. In this paper, we shall consider the Weyl-Lanczos equation (8).

In 1962 Lanczos proved the existence of the tensor L_{ijk} as a potential to the Weyl tensor C_{ijkl}, since then this potential has attracted the attention of a number of workers (cf., [2-8], [10-20], [22-23], [25], [27-28], [31-33]). The list of workers in this particular field is very large, here we have mentioned only few (for a detailed discussion of Lanczos potential , see [1]). The construction of L_{ijk}, for a given spacetime geometry, is equivalent to solving Weyl-Lanczos equation (8) with eqns (2-5) as constraints. Given the Weyl tensor, it is very difficult to construct the Lanczos potential by integrating directly the eqn (8) through tensorial approach. However, Newman-Penrose formalism ([29]) offers some simplifications.

It is known that most of the physically significant spacetime solutions are of Petrov type D. Some of the familiar members of this class are Schwarzchild, Riessner-Nördstrom, Kerr, Kerr-Newman, Vaidya and Gödel metrics. In this paper, we shall obtain the Lanczos potential for a number of Petrov type D metrics using the prescription given by Ahasn and Bilal [2]; which in turn leads to the solution of Weyl-Lanczos equations.

2. Petrov D spacetimes

Using Newman-Penrose formalism, a possible general solution of Weyl-Lanczos equations for Petrov type D spacetimes is given by ([2])

$$L_0 = \kappa\ ,\ L_1 = \frac{1}{3}\rho$$

$$L_2 = \frac{1}{3}\pi \ , \ L_3 = \lambda$$
$$L_4 = \sigma \ , \ L_5 = -\frac{1}{3}\tau \tag{9}$$
$$L_6 = -\frac{1}{3}\mu \ , \ L_7 = \nu$$

But for Petrov type D fields, Goldberg-Sachs theorem demands that $\kappa = \sigma = \nu = \lambda = 0$ and eqn (9) thus reduces to

$$L_0 \ = \ L_3 \ = \ L_4 \ = \ L_7 \ = \ 0$$
$$L_1 = \frac{1}{3}\rho, \ L_2 = \frac{1}{3}\pi, \ L_5 = -\frac{1}{3}\tau, \ L_6 = -\frac{1}{3}\mu \tag{10}$$

Therefore, if L_i $(i = 0, 1, ..., 7)$ [from eqns (10)] are known, then from the completeness relation

$$L_{ijk} = M_{ijk} + \bar{M}_{ijk} \tag{11}$$

between the Lanczos tensor L_{ijk} and Lanczos scalars L_i we can construct the Lanczos spin tensor which in turn generates the gravitational field (the Weyl tensor) through Weyl-Lanczos eqns (8) where

$$M_{ijk} = L_0 U_{ij} n_k + L_1(W_{ij} n_k - U_{ij} m_k) + L_2(V_{ij} n_k - W_{ij} m_k) - L_3 V_{ij} m_k$$
$$-L_4 U_{ij} \bar{m}_k + L_5(U_{ij} l_k - W_{ij} \bar{m}_k) + L_6(W_{ij} l_k - V_{ij} \bar{m}_k) + L_7 V_{ij} l_k \tag{12}$$

and

$$U_{ij} = -n_i \bar{m}_j + n_j \bar{m}_i \ , \ V_{ij} = l_i m_j - l_j m_i$$
$$W_{ij} = l_i n_j - l_j n_i + m_i \bar{m}_j - m_j \bar{m}_i \tag{13}$$

Here we shall find the Lanczos potential for some well known metrics using eqn (10). In each case, we shall write down the null tetrad for the metric under consideration, its non-vanishing spin-coefficients, intrinsic derivatives etc. and then the Lanczos scalars.

2.1. Kerr-Newman black hole

Newman et al ([30]) have obtained a solution of Einstein-Maxwell equations which, in (r, θ, ϕ, u) coordinates, is described by the metric ([24])

$$ds^2 = [1 - A^{-2}(2mr - e^2)]du^2 + 2dudr + 2aA^{-2}(2mr - e^2)\sin^2\theta dud\phi$$
$$-2a\sin^2\theta drd\phi - A^2 d\theta^2 - [(r^2 + a^2)^2 - Ba^2\sin^2\theta]A^{-2}\sin^2\theta d\phi^2 \tag{14}$$

where $A^2 = r^2 + a^2\cos^2\theta$ and $B = r^2 - 2mr + a^2 + e^2$. The solution (14) represents the exterior gravitational field of a charged rotating mass and contains three parameters - m (mass), e (charge) and a (angular momentum per unit mass).

The metric (14) is known as Kerr-Newman solution and defines a black hole with an event horizon only when $a^2 + e^2 \leq m^2$.

It is known that rotating black holes are formed due to gravitational collapse of a massive spinning star or from the collapse of a collection of stars or gas with a total non-zero angular momentum. Since most of the stars rotate, it is expected that most of the black holes in nature are rotating and thus Kerr-Newman solution represents the gravitational field outside a charged rotating black hole (see also [9]).

The null tetrad vectors for the metric (14) are

$$l^i = \delta^i_2,\ n^i = \frac{1}{A^2}\Big[(r^2+a^2)\delta^i_1 - \frac{B}{2}\delta^i_2 + a\delta^i_4\Big],$$
$$m^i = \frac{1}{C\sqrt{2}}\Big[ia\sin\theta\delta^i_1 + \delta^i_3 + i\csc\theta\delta^i_4\Big] \tag{15}$$

where $C = r + ia\cos\theta$. The non-zero spin-coefficients are given by

$$\rho = -\bar{C}^{-1},\ \mu = -\frac{B}{2\bar{C}A^2},\ \alpha = \frac{2ia - C\cos\theta}{2\sqrt{2}(\bar{C}\bar{C}\sin\theta)},\ \beta = \frac{\cot\theta}{2\sqrt{2}C}$$
$$\tau = -\frac{ia\sin\theta}{\sqrt{2}A^2},\ \pi = \frac{ia\sin\theta}{\sqrt{2}C\bar{C}},\ \gamma = \frac{(r-m)\bar{C} - B}{2\bar{C}A^2} \tag{16}$$

while the non-zero components of Weyl and Maxwell scalars, respectively, are

$$\Psi_2 = -\frac{m}{\bar{C}^3} + \frac{e^2}{C\bar{C}^3} \text{ and } \phi_1 = \frac{e}{\sqrt{2}\bar{C}\bar{C}} \tag{17}$$

which shows that Kerr-Newman solution (14) is of Petrov type D with non-null electromagnetic field.

Therefore, from eqns (10) and (16), the Lanczos scalars for the charged rotating black hole are given by

$$L_0 = L_3 = L_4 = L_7 = 0$$
$$L_1 = -\frac{1}{3\bar{C}},\ L_2 = \frac{ia\sin\theta}{3\sqrt{2}\bar{C}\bar{C}},\ L_5 = \frac{ia\sin\theta}{3\sqrt{2}C},\ L_6 = \frac{B}{6\bar{C}A^2} \tag{18}$$

2.2. Kerr black hole

When $e = 0$ and $a \neq 0$, eqn (14) reduces to Kerr solution (a rotating black hole) which can be expressed as

$$ds^2 = \Big[1 - \frac{2mr}{r^2 + a^2\cos\theta}\Big]du^2 + 2dudr + \Big[\frac{4amr\sin^2\theta}{r^2+a^2\cos\theta}\Big]dud\phi - 2a\sin^2\theta drd\phi$$

$$-(r^2+a^2\cos^2\theta)d\theta^2-\Big[\frac{(r^2+a^2)^2-(r^2-2mr+a^2)a^2\sin^2\theta}{r^2+a^2\cos^2\theta}\Big]\sin^2\theta d\phi^2 \quad (19)$$

The non-zero spin-coefficients are given by

$$\rho=-\bar{C}^{-1},\ \mu=-\frac{2mr-r^2-a^2}{2\bar{C}A^2},\ \alpha=\frac{2ia-C\cos\theta}{2\sqrt{2}(\bar{C}\bar{C}\sin\theta)},\ \beta=\frac{\cot\theta}{2\sqrt{2}C}$$

$$\tau=-\frac{ia\sin\theta}{\sqrt{2}A^2},\ \pi=\frac{ia\sin\theta}{\sqrt{2}C\bar{C}},\ \gamma=\frac{(r-m)\bar{C}-(r^2-2mr+a^2)}{2\bar{C}A^2} \quad (20)$$

where $C=r+ia\cos\theta$ and $A^2=r^2+a^2\cos\theta$.
The non-zero components of Weyl scalar is

$$\Psi_2=-\frac{m}{\bar{C}^3} \quad (21)$$

which shows that Kerr metric (19) is of Petrov type D. Hence, using eqn (10), the Lanczos scalars for Kerr black hole are given by

$$L_0=L_3=L_4=L_7=0$$

$$L_1=\frac{1}{3}\rho,\ L_2=\frac{1}{3}\pi,\ L_5=-\frac{1}{3}\tau,\ L_6=-\frac{1}{3}\mu \quad (22)$$

where the spin-coefficients ρ,π,τ and μ are given by eqn (20).

It may be noted that Petrov type D fields have only Coulomb component of the gravitational field and thus the Lanczos scalars, given by eqn (22), can act as the potential of the gravitational field of Kerr black hole (in analogy with the electromagnetism).

2.3. Reissner-Nördstrom metric

If the angular momentum a is zero, then the line-element (14) reduces to Reissner-Nördstrom solution given by

$$ds^2=\Big(1-\frac{2m}{r}+\frac{e^2}{r^2}\Big)du^2+2dudr-r^2(d\theta^2+\sin^2\theta d\phi^2) \quad (23)$$

The null tetrad for the metric (23) can be taken as ([24])

$$l_i=\delta^i_2,\ n^i=\delta^i_1-\frac{1}{2}\Big(1-\frac{2m}{r}+\frac{e^2}{r^2}\Big)\delta^i_2,\ m^i=\frac{1}{r\sqrt{2}}\Big(\delta^i_3+i\csc\theta\delta^i_4\Big) \quad (24)$$

so that the non-zero spin-coefficients are

$$\rho=-\frac{1}{r},\ \mu=-\frac{1}{2r}\Big(1-\frac{2m}{r}+\frac{e^2}{r^2}\Big),$$

$$\alpha=-\beta=\frac{1}{2\sqrt{2}r}\cot\theta,\ \gamma=\frac{1}{2r^3}(mr-e^2) \quad (25)$$

and the non-zero components of Weyl and Maxwell scalars are, respectively, given by

$$\Psi_2 = \frac{e^2}{r^4} - \frac{m}{r^3},\ \phi_1 = \frac{e}{r^2\sqrt{2}} \tag{26}$$

Hence following the prescription (10), the Lanczos scalars for the Reissner-Nördstrom metric are

$$L_0 = L_2 = L_3 = L_4 = L_5 = L_7 = 0$$

$$L_1 = -\frac{1}{3r},\ L_6 = \frac{1}{6r}\Big(1 - \frac{2m}{r} + \frac{e^2}{r}\Big) \tag{27}$$

which clearly indicates that the Lanczos potential for Reissner-Nördstrom solution depends upon the radial coordinate r, as well as on mass m and charge e.

2.4. quad Schwarzchild exterior solution

Consider the Schwarzchild metric as

$$ds^2 = -\Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta^2 d\phi^2) + \Big(1 - \frac{2m}{r}\Big)dt^2 \tag{28}$$

Using Kinnersley null tetrad ([15])

$$l^i = \frac{1}{r^2 - 2mr}(r^2,\ r^2 - 2mr,\ 0,\ 0),\ n^i = \frac{1}{2r^2}(r^2,\ 2mr - r^2,\ 0,\ 0),$$

$$m^i = \frac{1}{r\sqrt{2}}(0,\ 0,\ ,\ 1,\ i\csc\theta) \tag{29}$$

the non-vanishing spin-coefficients are given by

$$\rho = -\frac{1}{r},\ \mu = -\frac{1}{2r}\Big(1 - \frac{2m}{r}\Big),\ \alpha = -\beta = -\frac{1}{2\sqrt{2}r}\cot\theta,\ \gamma = \frac{m}{2r^2} \tag{30}$$

while the non-zero components of Weyl is

$$\Psi_2 = -\frac{m}{r^3} \tag{31}$$

The intrinsic derivatives used here are given by

$$D = l^i\nabla_i, = \frac{r^2}{A}\frac{\partial}{\partial t} + \frac{\partial}{\partial r},\ \triangle = n^i\nabla_i = \frac{1}{2}\frac{\partial}{\partial t} - \frac{A}{2r^2}\frac{\partial}{\partial r},$$

$$\delta = m^i\nabla_i = \frac{1}{r\sqrt{2}}\Big(\frac{\partial}{\partial\theta} + icsc\theta\frac{\partial}{\partial\phi}\Big)$$

where $A = r^2 - 2mr$.

From eqn (31), it may be noted that the Schwarzchild exterior solution is of Petrov type D, and thus using eqn (10), the Lanczos scalars are

$$L_0 = L_2 = L_3 = L_4 = L_5 = L_7 = 0,$$
$$L_1 = -\frac{1}{3r},\ L_6 = \frac{1}{6r}(1 - \frac{2m}{r}) \tag{32}$$

Consider now the Schwarzchild solution in null coordinates $x^i = (u, r, \theta, \phi)$ as

$$ds^2 = \Big(1 - \frac{2m}{r}\Big)du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{33}$$

and taking the intrinsic derivatives as ([33])

$$D = \frac{\partial}{\partial r},\ \triangle = \frac{\partial}{\partial u} - \frac{1}{2}\Big(1 - \frac{2m}{r}\Big)\frac{\partial}{\partial r},\ \delta = \frac{1}{r\sqrt{2}}\Big(\frac{\partial}{\partial\theta} + icsc\theta\frac{\partial}{\partial\phi}\Big) \tag{34}$$

the non-zero spin-coefficients are given by

$$\rho = -\frac{1}{r},\ \mu = -\frac{1}{2r},\ \alpha = \frac{\alpha^\circ}{r},\ \beta = -\frac{\bar{\alpha}^\circ}{r},\ \gamma = -\frac{1}{2r} + \frac{m}{r^2} \tag{35}$$

where $\alpha^\circ = \bar{\alpha}^\circ = -\frac{\cot\theta}{\sqrt{2}}$. Thus, from eqn (10), the Lanczos scalars for Schwarzchild solution in null coordinates are given by

$$L_0 = L_2 = L_3 = L_4 = L_5 = L_7 = 0,\ 2L_6 = -L_1 = \frac{1}{3}(\frac{1}{r}) \tag{36}$$

Remark. It may be noted from eqn (32) that the Lanczos scalars depend upon two parameters r (the radial coordinate) and m (the mass), while eqn (36) shows that the Lanczos scalars depend only on radial coordinate r. This clearly indicates the non-uniqueness character of Lanczos potential and this feature of Lanczos potential is in close analogy with that of the potential for electromagnetic field. Moreover, eqn (36) tells us that Lanczos scalars are inversely proportional to the radial distance. The Vaidya's metric, recently studied by Hasmani and Panchal ([23]), also exhibit the same feature. Moreover, since Petrov type D fields have only Coulomb component Ψ_2 of the gravitational field with l^i and n^i as the propagation vectors, therefore Lanczos scalars can act as the potential of the gravitational field; and thus justifying the name - the *Lanczos potential.*

2.5. Kantowski-Sachs solution

A solution of Einstein field equations without cosmological constant for dust particles was given by Kantowski and Sachs ([26]) and the metric in spherical coordinates (r, θ, ϕ, t) is given by

$$ds^2 = dt^2 - X^2(t)dr^2 - Y^2(t)[d\theta^2 + \sin^2\theta d\phi^2] \tag{23}$$

The null tetrad vectors for this metric are ([24])

$$l^i = \frac{1}{\sqrt{2}}(\delta_4^i + X^{-1}\delta_1^i),\ n^i = \frac{1}{\sqrt{2}}(\delta_4^i - X^{-1}\delta_1^i),\ m^i = \frac{1}{Y\sqrt{2}}(\delta_2^i + i\sin^{-1}\theta\delta_3^i) \tag{24}$$

The non vanishing spin-coefficients are given as

$$\rho = -\mu = -\frac{Y'}{Y\sqrt{2}},\ \alpha = -\beta = \frac{\cot\theta}{2\sqrt{2}Y},\ \epsilon = -\gamma = \frac{X'}{2\sqrt{2}X} \tag{25}$$

and the non-zero components of Weyl scalar is

$$\Psi_2 = \frac{1}{3}\Big(\frac{Y''}{Y} - \frac{X''}{X}\Big)$$

where a dash denotes the differentiation with respect to t.

Therefore, from eqn (10), the Lanczos scalars for Kantowski-Sachs metric are

$$L_0 = L_2 = L_3 = L_4 = L_5 = L_7 = 0,\ L_1 = L_6 = -\frac{Y'}{3Y\sqrt{2}} \tag{26}$$

which shows that the Lanczos scalars depends only on time as Y is a function of time t.

3. Conclusion

The Lanczos potential for some well known solutions of Einstein and Einstein-Maxwell equations have been obtained using the techniques of Newman-Penrose formalisms. It has been observed that the Lanczos scalars can be expressed in terms of the spin-coefficients, and our conjecture is that it shall occur in any Petrov type if we select an appropriate null tetrad. Moreover, since Lanczos spin tensor is a geometrical object of spacetime therefore it can be interpreted physically; and an attempt has been made to assign a possible physical meaning to this tensor. Thus, for example, for Schwarzchild exterior solution the Lanczos scalars are inversely proportional to the radial distance. The Vaidya metric also exhibit the same feature. Moreover, since Petrov type D fields have only Coulomb component Ψ_2 of the gravitational field with l^i and n^i as the propagation vectors therefore Lanczos scalars can act as the potential of the gravitational field; and thus justifying the name - *the Lanczos potential*. The non-uniqueness character of Lanczos potential has also been established and it is seen that this character can be achieved by the different choices of the tetrad vectors. This non-uniqueness property of the Lanczos potential is in close analogy with the potential of the electromagnetic field.

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