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Finslerian Subspaces given by Generalized Conformal $\beta-$Change

Mallikarjun Kumbar and Narasimhamurthy S. K.*

Department of Mathematics,
Mahantswamy Arts, Science and Commerce College, Haunsbhavi - 581109,
Haveri, Karnataka, India
*Department of P. G. Studies and Research in Mathematics,
Kuvempu University, Shankaraghatta - 577451,
Shimoga, Karnataka, India
e-mail: mallikarjunykumbar@gmail.com, nmurthysk@gmail.com
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Abstract

In the present paper, we study Finslerian subspace F^m of F^n and another Finslerian subspace $\bar{F}^m$ of the $\bar{F}^n$ subjected to the generalized conformal β-change is totally geodesic and totally h-autoparallel.

Key Words: Finsler space, Finsler subspace, Generalized β-conformal change, β-conformal change, β-change, Conformal change.
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1. Introduction

In 1976, M. Hashiguchi [1] studied the conformal change of Finsler metrics, namely, $\bar{L} = e^{\sigma(x)}L$. In particular, he also dealt with the special conformal transformation named C-conformal transformation. This change has been studied by H. Izumi[2], V. K. Kropina[3]. In 2008, S. Abed ([4], [5]) introduced the transformation $\bar{L} = e^{\sigma(x)}L + \beta$, thus generalizing the conformal, Randers and generalized Randers changes. Moreover, he established the relationships between some important tensors associated with (M, L) and the corresponding tensors associated with $(M, \bar{L})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M, L) and the transformed Cartan connection associated with $(M, \bar{L})$.

In this paper, we deal with a general change of Finsler metrics defined by:

$$L(x,y) \to \bar{L}(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y))$$

where f is a positively homogeneous function of degree one in $\bar{L} := e^{\sigma}L$ and β. This change will be referred to as a generalized β-conformal change. It is clear that this change is a generalization of the above mentioned changes and deals simultaneously with β-change and conformal change. It combines also the special case of Shibata ($\bar{L} = f(L,\beta)$) and that of Abed ($\bar{L} = e^{\sigma}L, \beta$) .

In 1984, C. Shibata [6] studied β-change of Finsler metrics and discussed certain invariant tensors under such a change. In 1979, Singh, et.al.[7] studied a Randers space $F^n(M, L(x,y) = (g_{ij}(x)y^iy^j)^{\frac{1}{2}} + b_i(x)y^i), n \geq 2$ which undergoes a change $L(x,y) \mapsto L^*(x,y) = L^2(x,y) + (\alpha_i(x)y^i)^2$.

In the present paper, we consider a general Finsler space $F^n(M,L)$ which undergoes conformal and β-change, that is $L(x,y) \to \bar{L}(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y))$ where $\beta(x,y) = b_i(x)y^i$ is a 1-form. We study Finslerian subspace $F^m = (M^m, \bar{L}(u,v))$ of F^n and another Finslerian subspace $\bar{F}^m = (M^m, \bar{L}(u,v))$ of the $\bar{F}^n$ subjected to the generalized conformal β-change. Further, we consider a Finsler subspace is totally geodesic and totally h-autoparallel. For the notations and terminology, we refer the reader to the books [8] and [9], the papers [6] by Shibata and [10] by Youssef.

2. Preliminaries

Let $F^n = (M,L), n \geq 2$ be an n-dimensional C^∞ Finsler manifold with fundamental function $L = L(x,y)$. Consider the following change of Finsler structures which will be referred to as a generalized β-conformal change:

$$L(x,y) \longrightarrow \bar{L}(x,y) = f(e^{\sigma(x)}L(x,y), \beta(x,y)), \tag{2.1}$$

where f is a positively homogeneous function of degree one in $e^{\sigma}L$ and 1-form β where, $\beta = b_i(x)dx^i$.

We define

$$f_1 := \frac{\partial f}{\partial \tilde{L}},\ f_2 := \frac{\partial f}{\partial \beta},\ f_{12} := \frac{\partial^2 f}{\partial \tilde{L}\partial \beta}, \ldots..,$$

where $\tilde{L} = e^{\sigma}L$.

The angular metric tensor $\bar{h}_{ij}$ of the space $\bar{F}^n$ is given by [10]

$$\bar{h}_{ij} = e^{\sigma}ph_{ij} + q_0m_im_j \tag{2.2}$$

where

$$\begin{array}{rcl} \{ \; p & = & ff_1/L, \quad q = ff_2, \quad q_0 = ff_{22}, \quad p_0 = f_2^2 + q_0, \quad q_{-1} = ff_{12}/L, \\ p_{-1} & = & q_{-1} + pf_2/f, \quad q_{-2} = f(e^\sigma f_{11} - f_1/L)/L^2, \quad p_{-2} = q_{-2} + e^\sigma p^2/f^2, \\ m_i & = & b_i - \beta y^i/L^2 \neq 0, \qquad \sigma_i = \partial_i \sigma. \; \} \end{array} \tag{2.3}$$

h_{ij} being the angular metric tensor of F^n. The fundamental metric tensor $\bar{g}_{ij}$ and its inverse $\bar{g}^{ij}$ of $\bar{F}^n$ are expressed as [10]

$$\bar{g}_{ij} = e^\sigma p g_{ij} + p_0 b_i b_j + e^\sigma p_{-1}(b_i y_j + b_j y_i) + e^\sigma p_{-2} y_i y_j, \tag{2.4}$$

$$\bar{g}^{ij} = (e^{-\sigma}/p) g^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j, \tag{2.5}$$

where

$$\begin{array}{rcl} \{ \; s_0 & = & e^{-\sigma} f^2 q_0/(\varepsilon p L^2), \quad s_{-1} = p_{-1} f^2/(\varepsilon p L^2), \\ s_{-2} & = & p_{-1}(e^\sigma m^2 p L^2 - b^2 f^2)/(\varepsilon p \beta L^2), \\ \varepsilon & = & f^2(e^\sigma p + m^2 q_0)/L^2 \neq 0, \quad m^2 = g^{ij} m_i m_j. \; \} \end{array} \tag{2.6}$$

g_{ij} and g^{ij} respectively being the metric tensor and inverse metric tensor of F^n. The Cartan tensor $\bar{C}_{ijk}$ and the associate Cartan tensor $\bar{C}^l_{ij}$ of $\bar{F}^n$ are given by the following expressions:

$$\bar{C}_{ijk} = e^\sigma p C_{ijk} + \frac{1}{2} e^\sigma p_{-1}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{1}{2} p_{02} m_i m_j m_k, \tag{2.7}$$

The $(h)hv$-torsion tensor $\bar{C}^l_{ij}$ is expressed in terms of C^l_{ij} as [10]

$$\bar{C}^l_{ij} = C^l_{ij} + M^l_{ij}, \tag{2.8}$$

where

$$\begin{array}{rcl} M^l_{ij} & = & \frac{1}{2p}[e^{-\sigma} m^l - p m^2(s_0 b^l + s_{-1} y^l)](e^\sigma p_{-1} h_{ij} + p_{02} m_i m_j) \\ & - & e^\sigma (s_0 b^l + s_{-1} y^l)(p C_{isj} b^s + p_{-1} m_i m_j) + \frac{p_{-1}}{2p}(h^l_i m_j + h^l_j m_i); \end{array} \tag{2.9}$$

$$h^i_j = g^{il} h_{lj}, \qquad p_{02} = \frac{\partial p_0}{\partial \beta}$$

C_{ijk} and C^l_{ij} respectively being the Cartan tensor and associate Cartan tensor of F^n. The spray coefficients $\bar{G}^i$ of $\bar{F}^n$ in terms of the spray coefficients G^i of F^n are expressed as [10]

$$\bar{G}^i = G^i + D^i, \tag{2.10}$$

where

$$\begin{aligned} D^i &= \frac{\sigma_0}{2p}\{[2p-\beta p_{-1}-e^{\sigma}p^2L^2s_{-2}-ps_{-1}(2e^{\sigma}p\beta+e^{\sigma}p_{-1}L^2m^2)]y^i \\ &- 2e^{\sigma}p^2\beta s_0b^i\}+\frac{q}{p}e^{-\sigma}F^i_0-\frac{1}{2}L^2\sigma^i+\frac{1}{2}(e^{\sigma}pE_{00}-2qF_{\beta 0} \\ &+ e^{\sigma}pL^2\sigma_{\beta})(s_0b^i+s_{-1}y^i); \\ E_{jk} &= (1/2)(b_{j|k}+b_{k|j}), \qquad F_{jk}=(1/2)(b_{j|k}-b_{k|j}), \qquad F^i_j=g^{ik}F_{kj} \end{aligned} \tag{2.11}$$

the symbol '|' denote the h-covariant derivative with respect to the Cartan connection $C\Gamma$ and the lower index '$_0$' (except in s_0) denote the contraction by y^i.

The relation between the coefficients $\bar{N}^i_j$ of Cartan nonlinear connection in $\bar{F}^n$ and the coefficients N^i_j of the corresponding Cartan nonlinear connection in F^n is given by [10]

$$\bar{N}^i_j=N^i_j+D^i_j, \tag{2.12}$$

where

$$\begin{aligned} D^i_j &= \frac{e^{-\sigma}}{p}A^i_j-(s_0b^i+s_{-1}y^i)A_{tj}b^t-(qb_{0|j}+e^{\sigma}pL^2\sigma_j). \\ & (s_{-1}b^i+s_{-2}y^i); \end{aligned} \tag{2.13}$$

$$\begin{aligned} A_{ij} &= E_{00}B_{ij}+F_{i0}Q_j+qF_{ij}+E_{j0}Q_i-2(e^{\sigma}pC_{sij}+V_{sij})D^s \\ &+ \frac{1}{2}\sigma_0[2e^{\sigma}pg_{ij}+2e^{\sigma}p_{-1}m_jy_i-2\beta B_{ij}+e^{\sigma}p_{-1}(b_iy_j-b_jy_i)] \\ &- \frac{1}{2}\sigma_i(e^{\sigma}L^2p_{-1}m_j+2e^{\sigma}py_j)+\frac{1}{2}\sigma_j(2e^{\sigma}py_i+e^{\sigma}L^2p_{-1}m_i); \end{aligned} \tag{2.14}$$

$$A^i_j=g^{li}A_{lj}, \qquad 2B_{ij}=e^{\sigma}p_{-1}h_{ij}+p_{02}m_im_j, \qquad Q_i=e^{\sigma}p_{-1}y_i+p_0b_i$$

The coefficients $\bar{F}^i_{jk}$ of Cartan connection $C\bar{\Gamma}$ in $\bar{F}^n$ and the coefficients F^i_{jk} of the corresponding Cartan connection $C\Gamma$ in F^n are related as [10]

$$\bar{F}^i_{jk}=F^i_{jk}+D^i_{jk}, \tag{2.15}$$

where

$$\begin{aligned} D^i_{jk} &= \{(e^{-\sigma}/p)g^{it}-(s_0b^i+s_{-1}y^i)b^t-(s_{-1}b^i+s_{-2}y^i)y^t\}\{F_{tk}Q_j \\ &+ F_{tj}Q_k+E_{jk}Q_t+\frac{1}{2}\Theta_{(j,k,t)}(2e^{\sigma}pC_{jkm}D^m_t+2V_{jkm}D^m_t \\ &- K_{jk}\sigma_t-2B_{jk}b_{0|t})\} \end{aligned} \tag{2.16}$$

$$V_{ijk} = \frac{1}{2}e^{\sigma}p_{-1}(h_{ij}m_k+h_{jk}m_i+h_{ki}m_j)+\frac{1}{2}p_{02}m_im_jm_k$$

$$\begin{aligned} K_{ij} &= A_1 g_{ij} + A_2 b_i b_j + A_3(b_i y_j + b_j y_i) + A_4 y_i y_j, \\ A_1 &= e^{\sigma}(2p - \beta p_{-1}), \quad A_2 = -\beta p_{02}, \quad A_3 = e^{\sigma} p_{-1} + (\beta^2/L^2) p_{02}, \\ A_4 &= e^{\sigma} p_{-2} - (\beta^3/L^4) p_{02}, \Theta_{(j,k,t)}\{A_{jkt}\} = A_{jkt} - A_{ktj} - A_{tjk}, \end{aligned}$$

The tensor D^i_{jk} has the properties:

$$D^i_{j0} = B^i_{j0} = D^i_j; \quad D^i_{00} = 2D^i, \quad where \quad B^i_{jk} = \partial_k D^i_j. \tag{2.17}$$

Lemma 2.1. [6]. If the covariant vector, the components $b_i(x)$ of which are coefficients of the one-form L, is parallel with respect to the Cartan connection $C\Gamma$ on F^n, then the difference tensor $D^i_{jk}(= \bar{F}^i_{jk} - F^i_{jk})$ vanishes.

3. Finslerian Subspaces given by Generalized conformal $\beta-$change

Let M^n be an n-dimensional smooth manifold and $F^n = (M^n, L)$ be an n-dimensional Finsler space equipped with a fundamental function $L(x,y)$ on M^n. Then the metric tensor $g_{ij}(x,y)$ and Cartans C-tensor $C_{ijk}(x,y)$ are given by

$$g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2, \quad C_{ijk} = (\partial g_{ij}/\partial y^k)/2,$$

and we can introduce in F^n the Cartan connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. An m-dimensional subspace M^m of the underlying smooth manifold M^n may be parametrically represented by the equation $x^i = x^i(u^\alpha)(i = 1, 2, ..., n)$, where u^α are Gaussian coordinates on M^m and Greek indices run from 1 to m. Here, we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank m. The following notations are also employed : $B^i_{\alpha\beta} := \partial^2 x^i/\partial u^\alpha \partial u^\beta, B^i_{0\beta} := v^\alpha B^i_{\alpha\beta}, B^{ij...}_{\alpha\beta....} := B^i_\alpha B^j_\beta.......$ If the supporting element y^i at a point (u^α) of M^m is assumed to be tangential to M^m, we may then write $y^i = B^i_\alpha(u) v^\alpha$, so that v^α is thought of as the supporting element of M^m at the point (u^α). Since the function $\underline{L}(u,v) := L(x(u), y(u,v))$ gives rise to a Finsler metric of M^m, we get an m-dimensional Finsler space $F^m = (M^m, \underline{L}(u,v))$.

At each point (u^α) of F^m, the unit normal vectors $N^i_\alpha(u,v)$ are defined by

$$g_{ij} B^i_\alpha N^j_a = 0, \quad g_{ij} N^i_a N^j_b = \delta_{ab} \quad (a, b, ... = m+1, ..., n). \tag{3.1}$$

If (B^α_i, N^a_i) is the inverse matrix of (B^i_α, N^i_a), we have

$$B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N^a_i = 0, \quad N^i_a B^\alpha_i = 0, \quad N^i_a N^b_i = \delta^b_a, \tag{3.2}$$

and further

$$B^i_\alpha B^\alpha_j + N^i_a N^a_j = \delta^i_j. \tag{3.3}$$

Making use of the inverse matrix $(g^{\alpha\beta})$ of $(g_{\alpha\beta})$, we get $B^\alpha_i = g^{\alpha\beta} g_{ij} B^j_\beta$. By (3.1) and (3.3), we also have $\delta_{ab} N^b_i = g_{ij} N^j_a$.

For the induced Cartan connection $C\Gamma = (F^\alpha_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ on F^m, the second fundamental h-tensor $H^a_{\alpha\beta}$ and the normal curvature vector H^a_α in a normal direction N^i_a are given by

$$\begin{aligned} H^a_{\alpha\beta} &= N^a_i (B^i_{\alpha\beta} + F^i_{jk} B^{jk}_{\alpha\beta}) + M^a_{\alpha b} H^b_\beta, \\ H^a_\alpha &= N^a_i (B^i_{0\alpha} + G^i_j B^j_\alpha), \end{aligned} \tag{3.4}$$

where $M^a_{\alpha b} := C^j_{ik} B^i_\alpha N^a_j N^k_b$ and $B^i_{0\alpha} = B^i_{\beta\alpha} v^\beta$. Contracting $H^a_{\beta\alpha}$ by v^β, we immediately get

$$H^a_{0\alpha} := H^a_{\beta\alpha} v^\beta = H^a_\alpha. \tag{3.5}$$

Lets introduce in $F^n = (M^n, \bar{L})$ the Cartan connection $C\bar{\Gamma} = (\bar{F}^i_{jk}, \bar{G}^i_j, \bar{C}^i_{jk})$ from a generalized conformal β-change of the metric.

We now consider a Finslerian subspace $F^m = (M^m, \bar{L}(u, v))$ of F^n and another Finslerian subspace $\bar{F}^m = (M^m, \bar{L}(u, v))$ of the $\bar{F}^n$ given by the generalized conformal β-change. Let N^i_a be unit normal vectors at each point of F^m, and (B^α_i, N^a_i) be the inverse matrix of (B^i_α, N^i_a). The functions $B^i_\alpha(u)$ may be considered as components of m linearly independent vectors tangent to F^m and they are invariant under the generalized conformal β-change. The unit normal vectors $\bar{N}^i_a(u, v)$ of $\bar{F}^m$ are uniquely determined by

$$\bar{g}_{ij} B^i_\alpha \bar{N}^j_a = 0, \quad \bar{g}_{ij} \bar{N}^i_a \bar{N}^j_b = \delta_{ab}. \tag{3.6}$$

The fundamental tensor $\bar{g}_{ij} = (\partial^2 \bar{L}^2 / \partial y^i \partial y^j)/2$ of the Finsler space $\bar{F}^n$ given by (2.3), (2.4).

Now contracting (3.1) by v^α, we immediately get

$$y_i N^i_a = 0 \tag{3.7}$$

Further contracting (2.4) by $N^i_a N^j_b$ and paying attention to (3.1), (3.6) and (3.7), we have

$$\bar{g}_{ij} N^i_a N^j_b = e^\sigma p \delta_{ab} + p_0 (b_i N^i_a)(b_j N^j_b). \tag{3.8}$$

Putting a = b, then we obtain

$$\bar{g}_{ij} (\pm N^i_a / \sqrt{e^\sigma p + p_0 (b_i N^i_a)^2})(\pm N^j_a / \sqrt{e^\sigma p + p_0 (b_i N^i_a)^2}) = 1, \tag{3.9}$$

provided $e^{\sigma}p + p_0(b_iN_a^i)^2 > 0$. Therefore we can put

$$\bar{N}_a^i = N_a^i/\sqrt{e^{\sigma}p + p_0(b_iN_a^i)^2}, \qquad (3.10)$$

where we have chosen the sign " + " in order to fix an orientation. On using (3.1) and (3.7), the first condition of (3.6) gives us

$$(b_iN_a^i)(p_0b_jB_\alpha^j + e^{\sigma}y_jB_\alpha^j) = 0. \qquad (3.11)$$

Now, assuming that $p_0b_jB_\alpha^j + e^{\sigma}p_{-1}y_jB_\alpha^j = 0$ and contracting this by v^α, we find $p_0\beta + e^{\sigma}p_{-1}L^2 = 0$. By (2.3) this equation lead us to $ff_\beta = 0$, where we have used $Lf_{L\beta} + \beta f_{\beta\beta} = 0$ and $Lf_L + \beta f_\beta = f$ owing to the homogeneity of f. Thus we have $f_\beta = 0$ because of $f \neq 0$. This fact means $\bar{L} = f(L)$ and contradicts the definition of a generalized conformal β-change of metric. Consequently (3.11) gives us

$$b_iN_a^i = 0. \qquad (3.12)$$

Therefore (3.10) is rewritten as

$$\bar{N}_a^i = N_a^i/\sqrt{e^{\sigma}p} \qquad (p > 0). \qquad (3.13)$$

and then it is clear $\bar{N}_a^i$ satisfies (3.6). Summarizing the above, we obtain

Theorem 3.1. For a field of linear frame $(B_1^i, ..., B_m^i, N_{m+1}^i, ..., N_n^i)$ of F^n, there exists a field of linear frame $(B_1^i, ..., B_m^i, \bar{N}_{m+1}^i, ..., \bar{N}_n^i)$ of $\bar{F}^n$ given by the generalized conformal β-change such that (3.6) is satisfied along $\bar{F}^m$, and then we get (3.12).

The quantities $\bar{B}_i^\alpha$ are uniquely defined along $\bar{F}^m$ by

$$\bar{B}_i^\alpha = \bar{g}^{\alpha\beta}\bar{g}_{ij}B_\beta^j, \qquad (3.14)$$

where $\bar{g}^{\alpha\beta}$ is the inverse matrix of $\bar{g}_{\alpha\beta}$. Let $(\bar{B}_i^\alpha, \bar{N}_i^a)$ be the inverse matrix of $(B_\alpha^i, \bar{N}_a^i)$, we have

$$B_\alpha^i\bar{B}_i^\beta = \delta_\alpha^\beta, \quad B_\alpha^i\bar{N}_i^a = 0, \quad \bar{N}_a^i\bar{B}_i^\alpha = 0, \quad \bar{N}_a^i\bar{N}_i^b = \delta_a^b, \qquad (3.15)$$

and further

$$B_\alpha^i\bar{B}_j^\alpha + \bar{N}_a^i\bar{N}_j^a = \delta_j^i. \qquad (3.16)$$

we also get $\delta_{ab}\bar{N}_i^b = \bar{g}_{ij}\bar{N}_a^j$, that is,

$$\bar{N}_i^a = \sqrt{e^{\sigma}p}N_i^a. \qquad (3.17)$$

Now assuming that the covector field $b_i(x)$ is gradient, we have from (2.11)

$$N_i^aD^i = 0. \qquad (3.18)$$

Differentiating (3.18) by y^j and contracting it by B^j_α, we get

$$N^a_i D^i_j B^j_\alpha = 0. \tag{3.19}$$

If each geodesic of F^m with respect to the induced metric is also a geodesic of F^n, then F^m is called totally geodesic. A totally geodesic subspace F^m is characterized by each $H^a_\alpha = 0$. From (3.4) and (3.17) we have

$$\bar{H}^a_\alpha = \sqrt{e^\sigma p}(H^a_\alpha + N^a_i D^i_j B^j_\alpha). \tag{3.20}$$

Thus from (3.19) we obtain $\bar{H}^a_\alpha = \sqrt{e^\sigma p} H^a_\alpha$. Hence we have

Theorem 3.2. Assume that the covector field $b_i(x)$ is gradient. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F^m}$ is totally geodesic.

From (3.4), (3.17) and Lemma 2.1, we have $\bar{H}^a_\alpha = \sqrt{e^\sigma p} H^a_\alpha$. Thus we obtain

Theorem 3.3. Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally geodesic, if and only if the subspace $\bar{F^m}$ is totally geodesic.

If each h-path of F^m with respect to the induced connection is also an h-path of F^n, then F^m is called totally h-autoparallel. A totally h-autoparallel subspace F^m is characterized by each $H^a_{\alpha\beta} = 0$. From (3.4), (3.5), (3.17) and Lemma 2.1, we obtain

Theorem 3.4. Let $b_i(x)$ be parallel with respect to $C\Gamma$ on F^n. Then the subspace F^m is totally h-autoparallel, if and only if the subspace $\bar{F^m}$ is totally h-autoparallel.

4. Conclusion

We have tried to generalize theorems of Finsler geometry and found that those not relying on the notion of translation may be successfully generalized.

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