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On LP-Sasakian Manifolds satisfying Certain Curvature Tensors

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Abstract

The object of the present paper is to study some curvature condition on LP-Sasakian manifolds which satisfy $P.\widetilde{W_2} = 0$, $\widetilde{W_2}.\widetilde{W_2} = 0$, $L.\widetilde{W_2} = 0$, $\widetilde{W_2}.L = 0$, $W_2.\widetilde{W_2} = 0$ and $\widetilde{W_2}.R = 0$.

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1. Introduction

In 1989, K. Matsumoto introduced the notion of LP-Sasakian manifold. Then I. Mihai and R. Rosca [3] introduced the same notion independently and they obtained several results on this manifold. LP-Sasakian manifold have also been studied by K. Matsumoto and I. Mihai [6], B. Prasad [1], U. C. De, K. Matsumoto and A. A. Shaikh [9], M. Trafdar and A. Bhattacharyya [7], Venkatesha and C. S. Bagewadi [10], Shaikh, Prakasha and Ahmad [11], Prasad and Haseeb [12], Berman [13] and others.

2. Preliminaries

A differentiable manifold M^n of dimension n is called LP-Sasakian manifold Matsumato [4, 6], if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2(X) = X + \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \quad D_X\xi = \phi X, \tag{2.4}$$

$$(D_X\eta)(Y) = g(\phi X, Y), \tag{2.5}$$

Let us put

$$F(X, Y) = g(\phi X, Y). \tag{2.6}$$

Then the tensor field F of (0,2) type is symmetric i.e.

$$F(X, Y) = F(Y, X), \tag{2.7}$$

$$(D_X\phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{2.8}$$

where D denotes the operator of covariant differention with respect to the Lorentzian metric g.

It can easily be seen that in an LP-Sasakian manifold the following relation hold:

$$\phi\xi = 0, \qquad \eta(\phi X) = 0. \tag{2.9}$$

Also, an LP-Sasakian manifold M^n are said to be Einstein and η-Einstein if its Ricci tensor S is of the form

$$\begin{aligned} &S(X, Y) = a'g(X, Y) \\ \text{and} \quad &S(X, Y) = a'\ g(X, Y) + b'\ \eta(X)\eta(Y), \end{aligned} \tag{2.10}$$

for any vector field X, Y where a', b' are function on M^n. Further on such a LP-Sasakian manifold with (ϕ, η, ξ, g) structure the following relation holds (Matsumoto and Mihai [6] and De, Matsumoto and Shaikh [9])

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.11}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(\xi, X)\xi = X + \eta(X), \tag{2.13}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.14}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.15}$$

$$Q\xi = (n-1)\xi \tag{2.16}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.17}$$

The notion of pseudo-W_2 curvature tensor $\widetilde{W_2}$ was given by (Prasad and Maurya [2])

$$\begin{aligned} \widetilde{W_2}(X, Y)Z =& aR(X, Y)Z + b[g(Y, Z)QX - g(X, Z)QY] - \\ & \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)X - g(X, Z)Y]. \end{aligned} \tag{2.18}$$

If $a = 1,\ b = -\frac{1}{n-1}$, then (2.18) takes the form

$$\begin{aligned}\widetilde{W_2}(X,Y)Z &= R(X,Y)Z - \frac{1}{n-1}[g(Y,Z)QX - g(X,Z)QY] \\ &= W_2(X,Y)Z.\end{aligned} \tag{2.19}$$

Thus W_2-curvature is particular case of $\widetilde{W_2}$-curvature, where a and b are real constant and R, Q and r are the Riemannian curvature tensor of type (1,3), the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and scalar curvature respectively. Prasad et. al [15] and Kumar [14] extended this notation on LP-Sasakian manifold and LP-Sasakian manifold with coefficient α.

On the other hand, the projective curvature tensor P and the concircular tensor L in a Riemannian manifold (M^n, g) defined by (Mishra [8])

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{2.20}$$

and

$$L(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{2.21}$$

Putting ξ for X in (2.18), (2.19) (2.20) and (2.21) and using (2.14),(2.15) and (2.16), we have

$$\widetilde{W_2}(\xi,Y)Z = k[g(Y,Z)\xi - \eta(Z)Y] + b[(n-1)g(Y,Z)\xi - \eta(Z)QY], \tag{2.22}$$

where $k = a - \frac{r}{n}\left[\frac{a}{n-1} + b\right]$.

$$W_2(\xi,Y)Z = [g(Y,Z)\xi - \eta(Z)Y] - \frac{1}{n-1}[(n-1)g(Y,Z)\xi - \eta(Z)QY], \tag{2.23}$$

$$P(\xi,Y)Z = g(Y,Z)\xi - \frac{1}{n-1}S(Y,Z)\xi, \tag{2.24}$$

$$L(\xi,Y)Z = \left[1 - \frac{r}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y]. \tag{2.25}$$

Now, we define $P(X,Y).\widetilde{W_2}$, $L(X,Y).\widetilde{W_2}$, $\widetilde{W_2}(X,Y).\widetilde{W_2}$, $\widetilde{W_2}(X,Y).L$, $W_2(X,Y).\widetilde{W_2}$ and $\widetilde{W_2}(X,Y).R$ as follows:

$$\begin{aligned}(P(X,Y).\widetilde{W_2})(U,V)W =& P(X,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(P(X,Y)U,V)W - \\ & \widetilde{W_2}(U,P(X,Y)V)W - \widetilde{W_2}(U,V)P(X,Y)W,\end{aligned} \tag{2.26}$$

$$\begin{aligned}(L(X,Y).\widetilde{W_2})(U,V)W =& L(X,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(L(X,Y)U,V)W - \\ & \widetilde{W_2}(U,L(X,Y)V)W - \widetilde{W_2}(U,V)L(X,Y)W,\end{aligned} \tag{2.27}$$

$$(\widetilde{W_2}(X,Y).\widetilde{W_2})(U,V)W = \widetilde{W_2}(X,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(\widetilde{W_2}(X,Y)U,V)W - \widetilde{W_2}(U,\widetilde{W_2}(X,Y)V)W - \widetilde{W_2}(U,V)\widetilde{W_2}(X,Y)W, \tag{2.28}$$

$$(W_2(X,Y).\widetilde{W_2})(U,V)W = W_2(X,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(W_2(X,Y)U,V)W - \widetilde{W_2}(U,W_2(X,Y)V)W - \widetilde{W_2}(U,V)W_2(X,Y)W, \tag{2.29}$$

$$(\widetilde{W_2}(X,Y).L)(U,V)W = \widetilde{W_2}(X,Y)L(U,V)W - L(\widetilde{W_2}(X,Y)U,V)W - L(U,\widetilde{W_2}(X,Y)V)W - L(U,V)\widetilde{W_2}(X,Y)W, \tag{2.30}$$

$$(\widetilde{W_2}(X,Y).R)(U,V)W = \widetilde{W_2}(X,Y)R(U,V)W - R(\widetilde{W_2}(X,Y)U,V)W - R(U,\widetilde{W_2}(X,Y)V)W - R(U,V)\widetilde{W_2}(X,Y)W. \tag{2.31}$$

3. LP-Sasakian manifold satisfying $P(\xi, Y).\widetilde{W_2}$

In this section we consider a LP-Sasakian manifold M^n satisfying the condition

$$P(\xi,Y).\widetilde{W_2} = 0. \tag{3.1}$$

From (2.26), we have

$$(P(\xi,Y).\widetilde{W_2})(U,V)W = P(\xi,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(P(\xi,Y)U,V)W - \widetilde{W_2}(U,P(\xi,Y)V)W - \widetilde{W_2}(U,V)P(\xi,Y)W,$$

Using (3.1), we have

$$P(\xi,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(P(\xi,Y)U,V)W - \widetilde{W_2}(U,P(\xi,Y)V)W - \widetilde{W_2}(U,V)P(\xi,Y)W = 0. \tag{3.2}$$

Taking the inner product with X and using (2.24) and (3.2), we get

$$\begin{aligned}&[g(Y,\widetilde{W_2}(U,V)W)\eta(X) - g(Y,U)g(\widetilde{W_2}(\xi,V)W,X) - \\ &g(Y,V)g(\widetilde{W_2}(U,\xi)W,X) - g(Y,W)g(\widetilde{W_2}(U,V)\xi,X)] - \\ &\frac{1}{n-1}[S(Y,\widetilde{W_2}(U,V)W)\eta(X) - S(Y,U)g(\widetilde{W_2}(\xi,V)W,X) \\ &- S(Y,V)g(\widetilde{W_2}(U,\xi)W,X) - S(Y,W)g(\widetilde{W_2}(U,V)\xi,X)] = 0.\end{aligned} \tag{3.3}$$

Taking ξ for U in (3.3) and using (2.4) and (2.15), we have

$$\begin{aligned}&[g(Y,\widetilde{W_2}(\xi,V)W)\eta(X)-g(Y,W)g(\widetilde{W_2}(\xi,V)\xi,X)]-\\&\frac{1}{n-1}[S(Y,\widetilde{W_2}(\xi,V)W)\eta(X)-S(Y,W)g(\widetilde{W_2}(\xi,V)\xi,X)]=0,\end{aligned}\tag{3.4}$$

Using (2.23) in (3.4), we have

$$\begin{aligned}&-k[\{g(Y,V)\eta(W)\eta(X)+g(Y,W)\eta(V)\eta(X)+g(Y,W)g(V,X)\}\\&-\frac{1}{n-1}\{S(Y,V)\eta(W)\eta(X)+S(Y,W)\eta(V)\eta(X)+S(Y,W)g(V,X)\}]\\&-b(n-1)[\{S(Y,V)\eta(W)\eta(X)+g(Y,W)\eta(V)\eta(X)+g(Y,W)S(V,X)\}\\&-\frac{1}{n-1}\{S(QY,V)\eta(W)\eta(X)+S(Y,W)\eta(V)\eta(X)+S(Y,W)g(V,X)\}]=0,\end{aligned}\tag{3.5}$$

where S(QY,Z)=S^2(Y,Z).
Putting e_i for X and W in (3.5), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over e_i, $1 \le i \le n$, we get

$$[r-n(n-1)][b(n-1)S(V,X)+kg(V,X)+\{k+b(n-1)\}\eta(V)\eta(X)]=0.\tag{3.6}$$

From (3.6), we have

$$\begin{aligned}&[r-n(n-1)]=0, \quad and \quad b(n-1)S(V,X)+kg(V,X)\\&+\{k+b(n-1)\}\eta(V)\eta(X)=0,\end{aligned}\tag{3.7}$$

respectively. In the case of $r-n(n-1)\neq 0$, if $b=0$, then from (3.7), we get

$$a[r-n(n-1)]=0.$$

But $r-n(n-1)\neq 0$, we get $a=0$.
This is the contradiction. Thus we find b≠0.
In view of (3.7), we have

$$\begin{aligned}S(V,X)=&\left[\frac{a}{b(1-n)}+\frac{r}{n}\left(\frac{a}{b(n-1)^2}\right)\right]g(V,X)+\\&\left[\frac{a}{b(1-n)}+\frac{r}{n}\left(\frac{a}{b(n-1)^2}\right)+1\right]\eta(V)\eta(X),\end{aligned}$$

which is an η-Einstein manifold.
Hence we have the following theorem:

Theorem 3.1. Let M^n be an n-dimensional ($n > 2$) LP-Sasakian manifold satisfying the condition $P(\xi, Y).\widetilde{W_2}$=0. Then M^n is a part of

(i) $r = n(n-1)$, that is, scalar curvature is constant or
(ii) an η-Einstein manifold, provided, $a \neq 0$, and $b \neq 0$.

4. LP-Sasakian manifold satisfying $\widetilde{W_2}(\xi, Y).\widetilde{W_2}$

In this section we consider a LP-Sasakian manifold M^n satisfying the condition

$$\widetilde{W_2}(\xi, Y).\widetilde{W_2} = 0. \tag{4.1}$$

From (2.28), we have

$$\begin{aligned}(\widetilde{W_2}(\xi, Y).\widetilde{W_2})(U, V)W =& \widetilde{W_2}(\xi, Y)\widetilde{W_2}(U, V)W - \widetilde{W_2}(\widetilde{W_2}(\xi, Y)U, V)W - \\ & \widetilde{W_2}(U, \widetilde{W_2}(\xi, Y)V)W - \widetilde{W_2}(U, V)\widetilde{W_2}(\xi, Y)W,\end{aligned}$$

Using (4.1), we have

$$\begin{aligned}&\widetilde{W_2}(\xi, Y)\widetilde{W_2}(U, V)W - \widetilde{W_2}(\widetilde{W_2}(\xi, Y)U, V)W - \\ &\widetilde{W_2}(U, \widetilde{W_2}(\xi, Y)V)W - \widetilde{W_2}(U, V)\widetilde{W_2}(\xi, Y)W = 0.\end{aligned} \tag{4.2}$$

Taking the inner product with X and putting ξ for U in (4.2), we have

$$\begin{aligned}&g(\widetilde{W_2}(\xi, Y)\widetilde{W_2}(\xi, V)W, X) - g(\widetilde{W_2}(\widetilde{W_2}(\xi, Y)\xi, V)W, X) - \\ &g(\widetilde{W_2}(\xi, \widetilde{W_2}(\xi, Y)V)W, , X) - g(\widetilde{W_2}(\xi, V)\widetilde{W_2}(\xi, Y)W, X) = 0.\end{aligned} \tag{4.3}$$

Putting e_i for X in (4.3), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over e_i, $1 \leq i \leq n$, we get

$$\begin{aligned}&g(\widetilde{W_2}(\xi, Y)\widetilde{W_2}(\xi, e_i)W, e_i) - g(\widetilde{W_2}(\widetilde{W_2}(\xi, Y)\xi, e_i)W, e_i) - \\ &g(\widetilde{W_2}(\xi, \widetilde{W_2}(\xi, Y)e_i)W, , e_i) - g(\widetilde{W_2}(\xi, e_i)\widetilde{W_2}(\xi, Y)W, e_i) = 0.\end{aligned} \tag{4.4}$$

In view of (2.18),(2.22) and (4.4), we have

$$\begin{aligned}&(a-b)\left[bS(QY, W) - \left(k - \frac{rb}{n}\right)S(Y, W)\right] \\ &+ (n-1)\left[k\{a + b(n-2)\} - b^2\{r - (n-1)\}\right]g(Y, W) = 0.\end{aligned} \tag{4.5}$$

If $b = 0$ and $k = a\left[1 - \frac{r}{n(n-1)}\right]$, then veiw of (4.5), we have

$$a\left[\frac{r}{n(n-1)} - 1\right][S(Y, W) - (n-1)g(Y, W)] = 0.$$

If $a \neq 0$, then

$$S(Y,W) = (n-1)g(Y,W), \quad or \quad r = n(n-1).$$

Hence we have the following theorem:

Theorem 4.1. Let M^n be an n-dimensional ($n > 2$) LP-Sasakian manifold satisfying the condition $\widetilde{W_2}(\xi,Y).\widetilde{W_2}=0$. Then we get

(i) if $b = 0$, then M^n is an Einstein manifold,
(ii) if $b \neq 0$, then we get

$$\left[bS(QY,W) - \left(\frac{k}{b} - \frac{r}{n}\right)S(Y,W)\right] - \frac{n-1}{(a-b)b}$$
$$\left[k\left\{a + b(n-2)\right\} - b^2\left\{r - (n-1)\right\}\right]g(Y,W) = 0.$$

5. LP-Sasakian manifold satisfying $L(\xi, Y).\widetilde{W_2}$

In this section we consider a LP-Sasakian manifold M^n satisfying the condition

$$L(\xi,Y).\widetilde{W_2} = 0. \tag{5.1}$$

From (2.27), we have

$$\begin{aligned}(L(\xi,Y).\widetilde{W_2})(U,V)W =& L(\xi,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(L(\xi,Y)U,V)W - \\ & \widetilde{W_2}(U,L(\xi,Y)V)W - \widetilde{W_2}(U,V)L(\xi,Y)W.\end{aligned}$$

Using (5.1), we have

$$\begin{aligned}& L(\xi,Y)\widetilde{W_2}(U,V)W - \widetilde{W_2}(L(\xi,Y)U,V)W - \\ & \widetilde{W_2}(U,L(\xi,Y)V)W - \widetilde{W_2}(U,V)L(\xi,Y)W = 0.\end{aligned} \tag{5.2}$$

Taking the inner product with X in (5.2) and using (2.25), we have

$$\begin{aligned}&\left[1 - \frac{r}{n(n-1)}\right][g(Y,\widetilde{W_2}(U,V)W)\eta(X) - \eta(\widetilde{W_2}(U,V)W)g(X,Y) \\ & - g(Y,U)g(\widetilde{W_2}(\xi,V)W,X) + \eta(U)g(\widetilde{W_2}(Y,V)W,X) - \\ & g(Y,V)g(\widetilde{W_2}(u,\xi)W,X) + \eta(V)g(\widetilde{W_2}(U,Y)W,X) - \\ & g(\widetilde{W_2}(U,V)\xi,X)g(Y,W) + g(\widetilde{W_2}(U,V)Y,X)\eta(W)] = 0.\end{aligned} \tag{5.3}$$

Again from (2.25), we have $r \neq n(n-1)$. Thus from (5.3), we have

$$\begin{aligned} &g(Y,\widetilde{W_2}(U,V)W)\eta(X) - \eta(\widetilde{W_2}(U,V)W)g(X,Y) \\ &- g(Y,U)g(\widetilde{W_2}(\xi,V)W,X) + \eta(U)g(\widetilde{W_2}(Y,V)W,X) - \\ &g(Y,V)g(\widetilde{W_2}(U,\xi)W,X) + \eta(V)g(\widetilde{W_2}(U,Y)W,X) - \\ &g(\widetilde{W_2}(U,V)\xi,X)g(Y,W) + g(\widetilde{W_2}(U,V)Y,X)\eta(W) = 0. \end{aligned} \tag{5.4}$$

Taking ξ for U in (5.4) and using (2.15), we have

$$\begin{aligned} &[g(V,W)g(X,Y) - g(Y,W)g(V,X)] + b[(n-1)g(V,W)g(X,Y) - \\ &S(Y,V)\eta(W)\eta(X) + (n-1)g(X,Y)\eta(V)\eta(W) + S(V,X)\eta(Y)\eta(W) \\ &- S(X,Y)\eta(V)\eta(W) - g(Y,W)S(V,X) + (n-1)g(V,Y)\eta(W)\eta(X) \\ &- S(V,X)\eta(Y)\eta(W)] - a' R(Y,V,W,X) - b[S(X,Y)g(V,W) \\ &- S(V,X)g(Y,W)] + \frac{r}{n}\left(\frac{a}{n-1} + b\right)[g(X,Y)g(V,W) \\ &- g(V,X)g(Y,W)] = 0. \end{aligned} \tag{5.5}$$

Putting e_i for Y and W in (5.5), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over e_i, $1 \leq i \leq n$, we get

$$\begin{aligned} (a-b)S(V,W) =& [a(n-1) + bn(n-1) - br]g(V,W) + \\ & b[n(n-1) - r]\eta(V)\eta(W). \end{aligned} \tag{5.6}$$

Hence we have the following theorem :

Theorem 5.1. An n-dimensional ($n > 2$) LP-Sasakian manifold M^n satisfying the condition $L(\xi,Y).\widetilde{W_2}$=0 is an η-Einstein manifold, provided $a - b \neq 0$ and $r \neq n(n-1)$.

6. LP-Sasakian manifold satisfying $\widetilde{W_2}(\xi, Y).L$

In this section we consider a LP-Sasakian manifold M^n satisfying the condition

$$\widetilde{W_2}(\xi,Y).L = 0. \tag{6.1}$$

From (2.30), we have

$$\begin{aligned} (\widetilde{W_2}(\xi,Y).L)(U,V)W =& \widetilde{W_2}(\xi,Y)L(U,V)W - L(\widetilde{W_2}(\xi,Y)U,V)W - \\ & L(U,\widetilde{W_2}(\xi,Y)V)W - L(U,V)\widetilde{W_2}(\xi,Y)W. \end{aligned}$$

Using (6.1), we have

$$\begin{aligned}&\widetilde{W_2}(\xi,Y)L(U,V)W-L(\widetilde{W_2}(\xi,Y)U,V)W-\\&L(U,\widetilde{W_2}(\xi,Y)V)W-L(U,V)\widetilde{W_2}(\xi,Y)W=0.\end{aligned}\tag{6.2}$$

Taking the inner product with X in (6.2) and using (2.22), we have

$$\begin{aligned}&(\widetilde{W_2}(\xi,Y)L(U,V)W,X)-g(L(\widetilde{W_2}(\xi,Y)U,V)W,X)-\\&g(L(U,\widetilde{W_2}(\xi,Y)V)W,X)-g(L(U,V)\widetilde{W_2}(\xi,Y)W,X)=0.\end{aligned}\tag{6.3}$$

Putting ξ for U in (6.3) and using (2.4),(2.1) and (2.25), we have

$$\begin{aligned}&\left[1-\frac{r}{n(n-1)}\right][k+b][g(V,W)g(X,Y)-g(Y,W)g(V,X)+\\&2g(V,W)S(X,Y)-(n-1)g(Y,V)\eta(W)\eta(X)+\eta(V)\eta(X)S(Y,W)-\\&(n-1)g(Y,W)\eta(V)\eta(X)-(n-1)g(Y,W)g(V,X)+\eta(W)\eta(X)S(V,Y)]-\\&k\left[g(R(Y,V)W,X)-\frac{r}{(n(n-1)}\{g(V,W)g(X,Y)-g(Y,W)g(V,X)\}\right]-\\&b\left[g(R(QY,V)W,X)-\frac{r}{(n(n-1)}\{g(V,W)S(X,Y)-S(Y,W)g(V,X)\}\right]=0.\end{aligned}\tag{6.4}$$

Putting e_i for X and V in (6.4), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over $e_i,\ 1\le i\le n$, we get

$$\begin{aligned}bS(QY,W)-&\left[\frac{br}{n-1}-k-b\right]S(Y,W)\\&-(n-1)\left[k+(n-1)b-\frac{br}{n}\right]g(Y,W)=0,\end{aligned}$$

when $b=0$, then above equation can be written as

$$k[S(Y,W)-(n-1)g(Y,W)]=0,$$

which means that $k[r-n(n-1)]=0$. From (2.25), we find $r\neq n(n-1)$. Thus we get $k=0$; namely $a=0$. This is the contradiction. Therefore we get $b\neq 0$ and

$$\begin{aligned}S(QY,W)=&\left[\frac{r}{n-1}-\frac{k}{b}-1\right]S(Y,W)-\\&(n-1)\left[\frac{r}{n}-\frac{k}{b}-(n-1)\right]g(Y,W).\end{aligned}\tag{6.5}$$

Hence we have the following theorem:

Theorem 6.1. In an n-dimensional ($n > 2$) LP-Sasakian manifold M^n satisfying the condition $\widetilde{W}_2(\xi, Y).L = 0$ holds on M^n, then the equation (6.5) is satisfied on M^n, provided $a \neq 0$ and $b \neq 0$.

7. LP-Sasakian manifold satisfying $W_2(\xi, Y).\widetilde{W}_2$

In this section we consider a LP-Sasakian manifold M^n satisfying the condition

$$W_2(\xi, Y).\widetilde{W}_2 = 0 \tag{7.1}$$

From (2.29), we have

$$\begin{aligned}(W_2(\xi, Y).\widetilde{W}_2)(U, V)W =& W_2(\xi, Y)\widetilde{W}_2(U, V)W - \widetilde{W}_2(W_2(\xi, Y)U, V)W - \\ & \widetilde{W}_2(U, W_2(\xi, Y)V)W - \widetilde{W}_2(U, V)W_2(\xi, Y)W,\end{aligned}$$

Using (7.1), we have

$$\begin{aligned}& W_2(\xi, Y)\widetilde{W}_2(U, V)W - \widetilde{W}_2(W_2(\xi, Y)U, V)W - \\ & \widetilde{W}_2(U, W_2(\xi, Y)V)W - \widetilde{W}_2(U, V)W_2(\xi, Y)W = 0.\end{aligned} \tag{7.2}$$

Taking the inner product with X and putting ξ for U, we have

$$\begin{aligned}& g(W_2(\xi, Y)\widetilde{W}_2(\xi, V)W, X) - g(\widetilde{W}_2(W_2(\xi, Y)\xi, V)W, X) - \\ & g(\widetilde{W}_2(\xi, W_2(\xi, Y)V)W, X) - W(\widetilde{W}_2(\xi, V)W_2(\xi, Y)W, X) = 0.\end{aligned} \tag{7.3}$$

Putting e_i for X and V in (7.3), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over e_i, $1 \leq i \leq n$, we get

$$(a - b)\left[S(QY, W) - \left\{\frac{r}{n} + (n-1)\right\} S(Y, W) + \frac{r}{n(n-1)} g(Y, W)\right] = 0.$$

Hence we have the following theorem:

Theorem 7.1. In an n-dimensional ($n > 2$) LP-Sasakian manifold satisfying the condition $W_2(\xi, Y).\widetilde{W}_2 = 0$, then we get

(i) if $a - b = 0$, or
(ii) if $a - b \neq 0$, then

$$S(QY, W) = \left\{\frac{r}{n} + (n-1)\right\} S(Y, W) - \frac{r}{n(1-n)} g(Y, W).$$

8. LP-Sasakian manifold satisfying $\widetilde{W_2}(\xi, Y).R$

In this section we consider a LP-Sasakian manifold M^n satisfying the condition

$$\widetilde{W_2}(\xi, Y).R = 0. \tag{8.1}$$

From (2.31) and (8.1), we have

$$\begin{aligned}(\widetilde{W_2}(\xi, Y).R)(U, V)W =& \widetilde{W_2}(\xi, Y)R(U, V)W - L(\widetilde{W_2}(\xi, Y)U, V)W - \\ & R(U, \widetilde{W_2}(\xi, Y)V)W - R(U, V)\widetilde{W_2}(\xi, Y)W,\end{aligned} \tag{8.2}$$

Taking the inner product with X and putting ξ for U in (8.2), we have

$$\begin{aligned}& k[g(V, W)g(X, Y) - g(R(Y, V)W, X) - g(Y, W)g(V, X) - g(V, X)(W)\eta(Y)] \\ & + b[g(V, W)S(X, Y) - (n-1)\eta(W)\eta(X)g(Y, V) - g(R(QY, V)W, X) \\ & + \eta(V)(X)S(Y, W) - (n-1)\eta(V)(X)g(Y, W) \\ & - (n-1)g(Y, W)g(V, X) + \eta(W)\eta(X)S(V, Y)] = 0.\end{aligned} \tag{8.3}$$

Putting e_i for X and V in (8.3), $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over e_i, $1 \leq i \leq n$, we get

$$bS(QY, W) + kS(Y, W) - (n-1)[k - (n-1)]g(Y, W) = 0. \tag{8.4}$$

If $b = 0$, then equation (8.4) will be

$$a[r - n(n-1)][S(Y, W) - (n-1)g(Y, W)] = 0,$$

Thus, we have

$$S(Y, W) = (n-1)g(Y, W).$$

Hence we have the following theorem:

Theorem 8.1. In an n-dimensional ($n > 2$) LP-Sasakian manifold M^n satisfying the condition $\widetilde{W_2}(\xi, Y).R = 0$, then we get

(i) if b =0, then the manifold is Einstein manifold, provided $a \neq 0$.

(ii) if $b \neq 0$, then

$$S(QY, W) = -\frac{k}{b}S(Y, W) + (n-1)[k - (n-1)]g(Y, W).$$

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