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A Study of Spacetimes with vanishing $M-$projective Curvature Tensor

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Abstract

In this paper we study about the $M-$projectively flat perfect fluid spacetime. First of all we showed that the Riemannian curvature tensor of an $M-$projectively flat spacetime is covariantly constant. Then we found the length of the Ricci operator in an $M-$projectively flat perfect fluid spacetime and proved that the isotropic pressure and entry density of an $M-$projectively flat perfect fluid spacetime satisfying Einsteins field equation with cosmological constant are constant. Then we showed that an $M-$projectively flat perfect fluid spacetime satisfying Einsteins field equation with cosmological constant and obeying the timelike convergence condition has positive isotropic pressure. Further we showed that the isotropic pressure and the energy density of an $M-$projectively flat perfect fluid spacetime satisfying Einsteins field equation with cosmological constant vanishes in a purely electromagnetic distribution. Lastly we showed that an $M-$projectively flat perfect fluid spacetime with the energy momentum tensor of an electromagnetic field such that the spacetime satisfies Einsteins field equation without cosmological constant is a Euclidean space.

Keywords: $M-$projectively flat perfect fluid spacetime, Riemannian curvature tensor, purely electromagnetic distribution.
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1. Introduction

This paper is dedicated to certain investigations in general relativity by the coordinate free method of differential geometry. In this method the spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with the Lorentzian metric g of signature $(-,+,+,+)$. The geometry of the Lorentzian manifold ([2]) begins with the study of the causal character of vectors of the manifold. This makes the Lorentzian manifolds a convenient choice in the study of general relativity. The Einsteins equations [4] (p.337), imply that the energy-momentum tensor is of vanishing divergence. We get this easily if the energy-momentum tensor is covariantly constant [15]. In the paper [15] M. C. Chaki and Sarbari Ray showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\Delta S = 0$, where S is the Ricci tensor of the spacetime. Many authors studied about spacetimes and its properties such as spacetimes with semisymmetric energy momentum tensor by De and Velimirovi [22], $M-$Projectively flat spacetimes by Zengin [7], pseudo Z symmetric spacetimes by Mantica and Suh [6], Mixed generalized quasi-Einstein manifold and some properties on it by D. Debnath, T. De and A. Bhattacharyya [1], On Ricci-symmetric mixed generalized quasi-Einstein spacetime by K. Chattopadhyay, N. Bhunia and A. Bhattacharyya [13], Concircular Curvature Tensor and Fluid Spacetimes by Z. Ahsan and S. A. Siddiqui in [23] and many more.

Let (M^n, g) be an n-dimensional differentiable manifold of class C^∞ with the metric tensor g and the Riemannian connection Δ. The M-projective curvature tensor of M^n defined by G. P. Pokhariyal and R.S. Mishra in 1971 (see [8]) in the following form

$$\begin{aligned} W(X,Y)Z = & R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1.1}$$

where R and S are the curvature tensor and the Ricci tensor of M^n, respectively. The tensor field W is known as the M-projective curvature tensor. Some authors studied the properties of this tensor (see [17], [16]). In 2010, S.K. Chaubey and R.H. Ojha studied about the $M-$projective curvature tensor of a Kenmotsu manifold (see [19]).

In this paper, the $M-$projectively flat perfect fluid spacetime will be investigated. And after that we will consider about a purely electromagnetic distribution and few results will be derived.

2. On the Riemannian curvature tensor of an M-projectively flat spacetime

The $M-$projective curvature tensor of a semi-Riemannian manifold satisfies the equation

$$\acute{W}(X,Y,Z,V) = \tilde{R}(X,Y,Z,V) - \frac{1}{2(n-1)}[S(Y,Z)g(X,V) - S(X,Z)g(Y,V)$$
$$+g(Y,Z)S(X,V) - g(X,Z)S(Y,V)]. \tag{2.1}$$

If it is an M-projectively flat spacetime then $n = 4$ and $W = 0$. Thus we get

$$\tilde{R}(X,Y,Z,V) = \frac{1}{6}[S(Y,Z)g(X,V) - S(X,Z)g(Y,V)$$
$$+g(Y,Z)S(X,V) - g(X,Z)S(Y,V)]. \tag{2.2}$$

Considering a frame field and taking contraction over X and V we get

$$S(Y,Z) = \frac{r}{4}g(Y,Z). \tag{2.3}$$

From equation (2.3) we observe that it's a manifold of constant curvature. The spaces of constant curvature play an important role in cosmology. The cosmological principle states that the simplest cosmological model is obtained by making the assumption that the universe is isotropic and homogeneous. In terms of Riemannian geometry it asserts that the three dimensional position space is a space of maximal symmetry [10], that is, a space of constant curvature whose curvature depends upon time. The cosmological solution of Einstein's equations which contain a three dimensional spacelike surface of a constant curvature are the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe (see [10] and [12]).

From equations (2.2) and (2.3) we get

$$\tilde{R}(X,Y,Z,V) = \frac{r}{12}[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)], \tag{2.4}$$

since r is constant thus we get

$$\Delta_Z \tilde{R} = 0. \tag{2.5}$$

Thus we obtain the following theorem:

Theorem 2.1. The Riemannian curvature tensor of an M-projectively flat spacetime is covariantly constant.

3. On $M-$projectively flat perfect fluid spacetimes

The general theory of relativity, which is a field theory of gravitation, is described by the Einstein's field equations. These equations whose fundamental constituent is the space-time metric g, are highly non-linear partial differential equations and therefore it is very difficult to obtain exact solutions. They become still more difficult to solve if the spacetime metric depends on all coordinates [10], [9]. This problem however, can be simplified to some extent if some geometric symmetry properties are assumed to be possessed by the metric tensor. There exists, by now, a reasonably large number of solutions of the Einstein's field equations possessing different symmetry structure [3]. The energy momentum tensor of a perfect fluid([4]) spacetime is given by

$$T(Y,Z) = (\sigma + p)A(Y)A(Z) + pg(Y,Z), \tag{3.1}$$

where σ is the energy density, p is the isotropic pressure and A is a non-zero 1-form such that $g(Y,\xi) = A(Y)$, for all Y. ξ is a unit timelike vector field which is the velocity vector field of the flow, i.e. $g(\xi,\xi) = -1$. Einstein's field equation with cosmological constant is given by

$$S(Y,Z) - \frac{r}{2}g(Y,Z) + \lambda g(Y,Z) = kT(Y,Z), \tag{3.2}$$

where r is the scalar curvature, λ is the cosmological constant and k is the gravitational constant, λ, $k \neq 0$.
Using equations (2.3), (3.1), (3.2) we get

$$(-\frac{r}{4} + \lambda - kp)g(Y,Z) = k(\sigma + p)A(Y)A(Z). \tag{3.3}$$

Taking contraction over Y and Z we get

$$r = 4\lambda + k(\sigma - 3p). \tag{3.4}$$

So from (2.3) and (3.4) we get

$$S(Y,Z) = (\lambda + \frac{k(\sigma - 3p)}{4})g(Y,Z). \tag{3.5}$$

Now we know that $g(QX,Y) = S(X,Y)$ and $S(QX,Y) = S^2(X,Y)$. Thus from (3.5) we get

$$S(QY,Z) = (\lambda + \frac{k(\sigma - 3p)}{4})^2 g(Y,Z). \tag{3.6}$$

Considering a frame field and taking contraction we get

$$||Q||^2 = \frac{1}{4}(4\lambda + k(\sigma - 3p))^2. \tag{3.7}$$

Thus we obtain the following theorem:

Theorem 3.1. In an M-projectively flat perfect fluid spacetime obeying Einstein's field equation with cosmological constant, the length of the Ricci operator is $\frac{1}{2}|4\lambda + k(\sigma - 3p)|$.

Now putting $Y = Z = \xi$ in (3.3) we get

$$r = 4(k\sigma + \lambda). \tag{3.8}$$

From (3.4) and (3.8) we obtain, $k(\sigma + p) = 0$, $k \neq 0$ will imply $\sigma + p = 0$, which means either $\sigma = p = 0$ (empty spacetime) or the perfect fluid satisfies the vacuumlike equation of state.
Thus we obtain the following theorem:

Theorem 3.2. An M-projectively flat perfect fluid spacetime obeying Einstein's field equation with cosmological constant is either a vacuum or satisfies the vacuumlike equation of state.

From equation (3.1), we obtain

$$T(Y, Z) = pg(Y, Z). \tag{3.9}$$

Since an M-projectively flat spacetime is Einstein, thus it's of constant scalar curvature r. Again λ , k are also constants, thus we get from (3.8) that σ is constant. Hence $p = -\sigma$ is also a constant. This condition has a special significance in the physics of spacetime. With this condition the fluid starts behaving like a cosmological constant(see [11]), which is also called as Phantom Barrier(see [20]). This causes inflation of the universe(see [14]).
Thus we obtain the following theorem:

Theorem 3.3. The isotropic pressure and energy density of an M-projectively flat perfect fluid spacetime satisfying Einsteins field equation with cosmological constant are constant. Furthermore the spacetime represents an inflation.

We know([18]) that the condition

$$S(X, X) > 0, \tag{3.10}$$

where S is a Ricci tensor of type $(0,2)$ and $g(X,X) < 0$, is called a timelike convergence condition.
Equations (3.1) and (3.2) imply

$$S(Y,Z) - \frac{r}{2}g(Y,Z) + \lambda g(Y,Z) = k[(\sigma + p)A(Y)A(Z) + pg(Y,Z)]. \quad (3.11)$$

Taking $Y = Z = \xi$ in (3.11) and using (3.8) we obtain

$$S(\xi,\xi) = -\lambda - k\sigma. \quad (3.12)$$

In 2001 Carmeli, M., Kuzmenko([5]) showed that the value of the cosmological constant $\lambda = 2.036 \cdot 10^{-35}s^{-2} > 0$. Thus from (3.10) and (3.12) we obtain $\sigma < -\frac{\lambda}{k}$. $\lambda, k > 0$ will imply

$$\sigma < 0. \quad (3.13)$$

Hence, $p = -\sigma > 0$.

Thus we obtain the following theorem:

Theorem 3.4. An M-projectively flat perfect fluid spacetime satisfying Einstein's field equation with cosmological constant and obeying the timelike convergence condition has positive isotropic pressure.

Again from (3.8) we get, $r > 0$, provided $-k\sigma < \lambda$.

Hence we obtain the following theorem:

Theorem 3.5. An M-projectively flat perfect fluid spacetime satisfying Einstein's field equation with cosmological constant and obeying the timelike convergence condition has positive scalar curvature provided $\lambda > -k\sigma$, where λ is the cosmological constant and k is the gravitational constant.

4. Perfect fluid spacetime in a purely electromagnetic distribution

Recently S. Mallick, P. Zhao and U. C. De showed in [21] that In a purely electromagnetic distribution for a perfect fluid spacetime satisfying Einstein's field equation without cosmological constant the trace of the energy-momentum tensor vanishes. Inspired by this from the last section we derive few results in a purely electromagnetic distribution.
From (3.9) after taking a frame field and doing contraction we get

$$\tau = 4p, \quad (4.1)$$

where, $\tau = trace\ T$. Again taking a frame field and doing contraction we get from (3.1)

$$\tau = 3p - \sigma. \tag{4.2}$$

Combining (4.1) and (4.2) we get, $\sigma = -p = -\frac{\tau}{4}$. Now since for a purely electromagnetic distribution $\tau = 0$.

Thus we can state:

Theorem 4.1. In a purely electromagnetic distribution the isotropic pressure and the energy density of a perfect fluid spacetime satisfying Einstein's field equation with cosmological constant vanishes.

Putting $\sigma = p = 0$ from (3.1) we get $T = 0$.

Hence we obtain the following corollary:

Corollary 4.1. In a purely electromagnetic distribution an M-projectively flat perfect fluid spacetime satisfying Einstein's field equation with cosmological constant is a vacuum.

Einstein's field equation without cosmological constant takes the form

$$S(Y,Z) - \frac{r}{2}g(Y,Z) = kT(Y,Z). \tag{4.3}$$

Taking a frame field and having contraction from (4.3) we get

$$r = -k\tau. \tag{4.4}$$

$\tau = 0$ implies $r = 0$.

So, from (2.4) we obtain

$$\tilde{R}(X,Y,Z,V) = 0. \tag{4.5}$$

Hence we obtain the following theorem:

Theorem 4.2. An M-projectively flat perfect fluid spacetime with the energy momentum tensor of an electromagnetic field such that the spacetime satisfies Einstein's field equation without cosmological constant is a Euclidean space.

Remark 4.1. This method shows a way to convert a Riemannian space to a Euclidean space.

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