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3-dimensional Locally $\phi-$Concircularly symmetric Lorentzian $\beta-$Kenmotsu Manifold

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Abstract

The present work deals with the study of 3-dimensional Locally $\phi-$ concircularly symmetric Lorentzian $\beta-$Kenmotsu manifold which generalizes the notion of locally concirculary-symmetric Lorentzian $\beta-$Kenmotsu manifold and obtain some interesting results. Also it is proved that a concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds is an Einstein manifold and Proved that if a concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds $(m^{2n+1}, g), n > 1$, has non zero constant sectional curvature, then it reduces to a concircularly locally $\phi-$symmetric manifold.

Keywords and Phrases: Concircularly $\phi-$symmetric manifold, concircularly $\phi-$recurrent manifold, Einstein manifold, Lorentzian $\beta-$Kenmotsu manifolds.
AMS Subject Classification: 53C25, 53C44, 53D10.

1. Introduction

A transformation of an n-dimensional Reimannian manifold M, which transforms every geodesic circle of M in to a geodesic circle, is called a concirculary transformation. A concircularly transformation is always a conformal transformation. Here a geodesic circle means a curve in M whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, that is the concircular geometry, is a generalization of inverse geometry in the sense that the change of metric is more general than that introduced by a circle preserving differomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor.

In this paper, we study a concircularly $\phi-$recurrent Lorentzian $\beta-$ Kenmotsu manifold which generalizes the notion of locally concirculary $\phi-$symmetric Lorentzian $\beta-$Kenmotsu manifold and obtain some interesting results. Again it is proved that a concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds is an Einstein manifold and proved that if a concircularly $\phi-$recurrent Lorentzian $\beta-$ Kenmotsu manifolds $(m^{2n+1}, g), n > 1$ has non-zero constant sectional curvature, then it reduces to a locally concircularly $\phi-$symmetric manifold. Further we study 3-dimentional locally $\phi-$concircularly symmetric Lorentzian $\beta-$Kenmotsu manifold.

2. Preliminaries

An $(2n+1)$ dimensional differentiable manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is called a Lorentzian $\beta-$Kenmotsu manifold with the structure (ϕ, ξ, η, g) where β is a smooth function on M if it admits a tensor field ϕ of type (1,1), a covariant vector field ξ, one 1 from η and a Lorentzian metric g which satisfy:

$$\phi^2 x = -x + \eta(x)\xi \tag{2.1}$$

$$\text{(a)} \quad \eta(\xi) = -1, \quad \text{(b)} \quad g(x, \xi) = \eta(x), \quad \text{(c)} \quad \eta(\phi x) = 0 \tag{2.2}$$

$$g(\phi x, \phi y) = g(x, y) + \eta(x)\eta(y) \tag{2.3}$$

$$(D_x, \phi)(y) = g(x, y)\xi - \eta(y)x \tag{2.4}$$

$$D_x\xi = \beta(x - \eta(x)\xi) \tag{2.5}$$

$$(D_x\eta)(Y) = \beta[(g(x, y)) - \eta(x)\eta(y)] \tag{2.6}$$

where D denotes the operator of covariant differentiation with respect to g. Also in Lorentzian $\beta-$Kenmotsu manifold, the following holds:

$$\eta(R(X, Y)Z) = g(R(X, Y)Z, \xi) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \tag{2.7}$$

$$R(X, Y)\xi = \beta^2[\eta(X)Y - \eta(Y)X] \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y)2n\beta^2\eta(X)\eta(Y) \tag{2.9}$$

$$S(x, \xi) = 2n\beta^2\eta(X) \tag{2.10}$$

For all vector fields X, Y, Z where S is the Ricci tensor of type (0,2) and R is the Riemannian Curvature tensor of the manifold.

Definition 2.1. A Lorentzian $\beta-$Kenmotsu manifold in said to be a locally $\phi-$symmetric manifold if.

$$\phi^2((D_wR)(X, Y)Z) = 0. \tag{2.11}$$

For all vector field X, Y, Z, W Orthogonal to ξ.

Definition 2.2. A Lorentzian $\beta-$Kenmotsu manifolds is said to be a locally concircularly $\phi-$symmetric manifold if.

$$\phi^2((D_wC)(X,Y)Z) = 0. \tag{2.12}$$

For all vector field X, Y, Z, W Orthogonal to ξ.

Definition 2.3. A Lorentzian $\beta-$Kenmotsu manifold is said to be concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmostu manifold if there exists a non -zero 1 from A such that.

$$\phi^2((D_wC)(X,Y)Z) = A(W)C(X,Y)Z, \tag{2.13}$$

for arbitrary vector fields X, Y, Z, W where C is a concircular curvature tensor given by

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}[g(Y,Z)Xg(X,Z)Y], \tag{2.14}$$

where R is the curvature tensor and r is the Scalar curvature. If the 1-form A vanishes, then the manifold reduces to a locally Concircularly $\phi-$ symmetric manifold.

3. Concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds

Let us consider a concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmostu manifolds. Then by virtue of (2.1) and (2.13) we get

$$(D_wC)(X,Y)Z + \eta((D_wC)(X,Y)Z)\xi = A(W)C(X,Y)Z, \tag{3.1}$$

from which it follows that

$$g((D_wC)(X,Y)Z,U) + \eta((D_wC)(X,Y)Z)\eta(U) = A(W)g(C(X,Y)Z,U) \tag{3.2}$$

Let $\{e_i\}, i = 1, 2, \ldots, 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold, Then putting $X = U = e_i$ in (3.2) and taking summation over i, $1 \leq i \leq 2n+1$, we get.

$$\begin{aligned}(D_wS)(Y,Z) = \; & -\frac{dr(W)}{2n+1}g(Y,Z) - \frac{dr(W)}{2n+1}g(Y,Z) - \eta(Y)\eta(Z) \\ & +A(W)[S(Y,Z) - \frac{r}{2n+1}g(Y,Z)].\end{aligned} \tag{3.3}$$

Replacing Z by ξ in (3.3) and using (2.5) and (2.10), we get

$$(D_wS)(Y,\xi) = \frac{dr(W)}{2n+1}\eta(y) - \frac{dr(W)}{2n(2n+1)}2\eta(y) + A(W)[-2n\beta - \frac{r}{2n+1}]\eta(Y). \tag{3.4}$$

Now, we have

$$(D_wS)(Y,\xi) = D_wS(Y,\xi) - S(D_wY,\xi) - S(Y,D_w\xi).$$

Using (2.5) and (2.10) in the above relation, it follows that

$$(D_wS)(Y,\xi) = -2n\beta^2 g(Y,W)\beta S(Y,W). \tag{3.5}$$

In view of (3.4) and (3.5), we get

$$\begin{aligned} S(Y,W) = & -2n\beta^2 g(Y,W) - \frac{dr(W)}{2n+1} + \eta(Y) + \frac{dr(W)}{2n(2n+1)}2\eta(Y) \\ & -A(W)[-2n\beta^2 - \frac{r}{2n+1}]\eta(Y). \end{aligned} \tag{3.6}$$

Replacing Y by ϕY and W by ϕW in (3.6) and using (2.3) and (2.9), we get

$$S(Y,W) = -2\beta^2 g(Y,W).$$

Hence we can state the following theorem

Theorem 3.1. A Concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds $M^{2n+1}, g)$ is an Einstein manifold. Now from (2.14) it follows that

$$(D_wC(X,Y)\xi = (D_wR)(X,Y)\xi - \frac{dr(W)}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \tag{3.7}$$

In view of (2.5) and (2.6) it can be easily seen that in a Lorentzian $\beta-$ Kenmotsu manifold the following relation holds:

$$(D_wR(X,Y)\xi = \beta^3[g(W,Y)X - g(W,X)Y + R(X,Y)W]. \tag{3.8}$$

Using (3.8) in (3.7), we get

$$\begin{aligned} (D_wC(X,Y)\xi \ &= \beta^3[g(W,Y)X - g(W,X)Y + R(X,Y)W] \\ & -\frac{dr(W)}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.9}$$

Using (2.7) in (3.9) we have

$$\begin{aligned} (D_wC(X,Y)\xi = & \beta^2(\beta-1)g(W,Y)X - \beta^2(\beta-1)g(W,X)Y \\ & -\frac{dr(W)}{2n(2n+1)}[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.10}$$

Now, we suppose that Lorentzian $\beta-$Kenmotsu $M^{2n+1}, g), n > 1$, is concircularly $\phi-$recurrent. Then from (3.1) and (3.10) it follows that

$$(D_wC)(X,Y)Z = -\beta^2(\beta-1)g(W,Y)g(X,Z) + \beta^2(\beta-1)g(W,X)g(Y,Z)$$
$$-\frac{dr(W)}{2n(2n+1)}[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)] + A(W)C(X,Y)Z.$$

Next we suppose that in a concircularly $\phi-$recurrent Lorentzian $\beta-$ Kenmotsu manifold, the sectional cur venture of a plane $\pi CTp(M)$ defined by

$$K_p(\pi) = g(R(X,Y)Y,X), \tag{3.11}$$

is a non zero constant K, where (X,Y) is any orthonormal basis of π. Then we have

$$g((D_zR)(X,Y)Y,X) = 0. \tag{3.12}$$

Again from (2.14) we get

$$(D_zC)(X,Y)Y = (D_zR)(X,Y)Y - \frac{dr(Z)}{2n(2n+1)}[g(Y,Y)X - g(X,Y)Y]. \tag{3.13}$$

In view of (3.12) it follow from (3.13) that

$$g((D_zC)(X,Y)Y,X) = 0. \tag{3.14}$$

By Virtue of (3.14) and (3.1) we have

$$g(D_zC)(X,Y)Y,\xi)\eta(X) = A(Z)g(C(X,Y)Y,X). \tag{3.15}$$

Since in a concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifold, the relation (3.12) holds good, using (3.10) in (3.15) we get

$$\begin{aligned}\eta(X)[\beta^2(\beta-1)g(Z.Y)g(X.Y) + \beta^2(\beta-1)g(Z.X)g(Y.Y)\\ -\frac{dr(Z)}{2n(2n+1)}[\{\eta(Y)G(X,Y) - \eta(X)g(Y,Y)\}\\ -A(Z)\{\beta^2 - \frac{r}{2n(2n+1)}\}\{g(Y,Y)\eta(X) - g(X,Y)\eta(Y)\}]\\ = A(z)[K\beta^2 - \frac{r}{2n(2n+1)}\{g.(Y,Y)g(X,X) - g(X,Y)g(X.Y)\}].\end{aligned} \tag{3.16}$$

Putting $Y = Z = \xi$ in (3.16) and using (2.2) and simplifying we get

$$n(\rho) = 0. \tag{3.17}$$

Hence from the relation $A(W) = -\eta(w)\eta(\rho)$, we get

$$A(W) = 0. \tag{3.18}$$

Using (3.18) in (3.1), we get

$$(\phi^2(D_wC)(X,Y)Z) = 0.$$

Hence we can state the following theorem:

Theorem 3.2. If a concircularly $\phi-$recurrent Lorentzian $\beta-$Kenmotsu manifolds $(M^{2n+1}, g), (n > 1)$, has a non-zero constant sectional curvature then it reduces to a locally concircularly $\phi-$symmetric manifold.

4. 3-dimensional Locally $\phi-$Concircularly symmetric Lorentzian $\beta-$ Kenmotsu manifold

In a 3-dimensional Lorentzian $\beta-$Kenmotsu manifold, concircular curvature tensor in given by

$$\begin{aligned} C(X,Y)Z = \frac{(r+4)}{2}[g(Y,Z)X - g(X,Z)Y]\frac{(r+6)}{2}[g(Y,Z)\eta(X)\xi \\ -g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ -\frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.1}$$

Taking Covariant differentiation of (4.1), we get

$$\begin{aligned} (D_wC)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)Xg(X,Z)Y] - \frac{dr(W)}{2}[g(Y,Z)\eta(X)\xi \\ g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] - \frac{(r+6)}{2}\times \\ [g(Y,Z)(D_w\eta)(X)\xi + g(Y,Z)\eta(X)(D_w\xi)g(X,Z)(D_w\eta)(Y)\xi] \\ -g(X,Z)\eta(Y)(D_w\xi) + (D_w\eta)(Y)\eta(Z)X + (D_w\eta)(Z)\eta(Y)X \\ -(D_w\eta)(X)\eta(Z)X - (D_w\eta)(Z)\eta(X)Y \\ -\frac{dr(W)}{6}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.2}$$

Taking X, Y, Z horizontal vector field and using (2.5) and (2.6), we get

$$\begin{aligned} (D_wC)(X,Y)Z = \frac{dr(W)}{2}[g(Y,Z)Xg(X,Z)Y] - \frac{(r+6)}{2}\beta[g(Y,Z) \\ g(X,W) - g(X,Z)g(Y,W)] + g(Y,Z)\eta(W)\eta(x) \\ +g(X,Z)\eta(W)\eta(Y)]\xi. \end{aligned} \tag{4.3}$$

From (4.3) it follows that

$$\phi^2(D_wC)(X,Y)Z) = \frac{dr(W)}{3}[g(Y,Z)\phi^2X - g(X,Z)\phi^2Y].$$

Now taking X, Y, Z horizontal vector field and using (2.1), (2.5) and (2.6), we get

$$\phi^2(D_wC)(X,Y)Z) = \frac{dr(W)}{3}[g(Y,Z)X - g(X,Z)Y].$$

Hence we state the following.

Theorem 4.1. A 3-dimensional Lorentzian $\beta-$Kenmotsu manifold is locally $\phi-$concircularly symmetric if and only if scalar curvature r is content.

In 1977 [3] has proved that

Corollary 4.1. A 3-dimensional Lorentzian $\beta-$Kenmotsu manifold is locally $\phi-$Symetric if and only if scalar curvature r is constant.

Using corollary (4.1) we state the following:

Theorem 4.2. A 3-dimensional Lorentzian $\beta-$Kenmotsu manifold is locally $\phi-$concircularly symmetric if and only if it is Locally $\phi-$Symmetric.

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