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Almost Kenmotsu Manifold Equipped with $M-$Projective Curvature Tensor

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Abstract

The object of the present paper is to study M-projective curvature tensor on almost Kenmotsu manifolds with characteristic vector field ξ belonging to some nullity distributions.

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1. Introduction

In 1972 Kenmotsu [8] introduced and studied a new class of almost contact metric manifolds called Kenmotsu manifolds. Almost contact metric manifold M^{2n+1} with 1-form η and fundamental 2-form Φ defined by $\Phi(X,Y) = g(X, \phi Y)$, where ϕ is a $(1,1)$ tensor field such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is called almost Kenmotsu manifold. The normality of an almost contact metric manifold is given by vanishing the $(1,2)$-type torsion tensor $N = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ [1]. According to [8], the normality of an almost Kenmotsu manifold is given by

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{1.1}$$

for every vector fields X, Y on M^{2n+1}. On the other hand, Gray [7] and Tanno [13] introduced the notion of k-nullity distribution, which is defined for any $p \in M^{2n+1}$ and $k \in \mathbb{R}$ as follows:

$$N_p(k) = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]\} \tag{1.2}$$

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for any $X, Y \in T_pM$, where T_pM^{2n+1} denotes the tangent vector space of M^{2n+1} at any point $p \in M^{22n+1}$ and R denotes the Riemannian curvature tensor of type $(1,3)$. Moreover, if k is a smooth function then the distribution is called generalized k-nullity distribution. Later Blair, Koufogiorgos and Papantoniou [2] introduced a generalized notion of the (k,μ)-nullity distribution on a contact metric manifold M^{2n+1} which is defined for any $p \in M^{2n+1}$ and $(k,\mu) \in \mathbb{R}^2$ as follows:

$$N_p(k,\mu) = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ +\mu[g(Y,Z)hX - g(X,Z)hY]\}, \qquad (1.3)$$

where $h = \frac{1}{2}\pounds_\xi\phi$ and $\pounds$ denotes the Lie differentiation. After this, Dileo and Pastore [5] introduced another generalized notion of the k-nullity distribution called $(k,\mu)'$-nullity distribution on an almost Kenmotsu manifold M^{2n+1} and is defined for any $p \in M^{2n+1}$ and $(k,\mu) \in \mathbb{R}^2$ as follows:

$$N_p(k,\mu)' = \{Z \in T_pM^{2n+1} : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ +\mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \qquad (1.4)$$

where $h' = h \circ \phi$. Recently, Dileo et al. ([5], [6], [7]), Wang et al. ([15], [16], [17]) and De et al. ([3], [9], [14]) obtained some important results on almost Kenmotsu manifolds with characteristic vector field ξ belonging to some nullity distributions. In this paper we investigate some results on almost Kenmotsu manifolds with M-projective curvature tensor W^* defined by [11]

$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \qquad (1.5)$$

where X, Y, Z are any vector fields and r is the scalar curvature. The paper is organized as follows: In Section 2, we give some basic formulas and properties of almost Kenmotsu manifolds according to Dileo and Pastore ([5], [7]). In Section 3, we study M-projectively semisymmetric almost Kenmotsu manifolds with ξ belonging to $(k,\mu)'$-nullity distribution. The Section 4 deals with the study of M-projectively Ricci-semisymmetric almost Kenmotsu manifolds with ξ belonging to $(k,\mu)'$-nullity distribution. The Section 5 and 6 are concerned with the study of M-projective curvature tensor on almost Kenmotsu manifolds with ξ belonging to (k,μ)-nullity distribution.

2. Almost Kenmotsu manifolds and nullity distributions

Let M^{2n+1} be an almost Kenmotsu manifold with structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a characteristic vector field and η is a 1-form and g is a Riemannian metric such that [1]

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad g(X,\xi) = \eta(X), \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields X, Y on M^{2n+1}. Let D be the distribution orthogonal to ξ and defined by $D = Ker(\eta) = Im(\phi)$. The two tensor fields $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot,\xi)\xi$ on an almost Kenmotsu manifold M^{2n+1} are symmetric and satisfy the following relations [10]

$$h\xi = 0, \; l\xi = 0, \; tr(h\phi) = 0, \; h\phi + \phi h = 0, \tag{2.3}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X, \tag{2.4}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

$$tr(l) = S(\xi,\xi) = g(Q\xi,\xi) = -2n - trh^2, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)(Y - \phi hY) - \eta(Y)(X - \phi hX) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{2.7}$$

for all vector fields X, Y on M^{2n+1}. Now we give some basic properties of almost Kenmotsu manifolds with ξ belonging to the $(k,\mu)'$-nullity distribution.
The $(1,1)$-type tensor field h' satisfies $h'\phi + \phi h' = 0$ and $h'\xi = 0$. Also it is known that

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{2.8}$$

For an almost Kenmotsu manifold, we have from (1.4)

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y] \quad and \tag{2.9}$$

$$R(\xi,X)Y = k[g(X,Y)\xi - \eta(Y)X] + \mu[g(h'X,Y)\xi - \eta(Y)h'X], \tag{2.10}$$

where $k, \mu \in \mathbb{R}$.
Contracting Y in (2.10), we get

$$S(X,\xi) = 2nk\eta(X). \tag{2.11}$$

Let $X \in D$ be the eigen vector of h' corresponding to the eigen value λ orthogonal to ξ. It follows from (2.8) that $\lambda^2 = -(k+1)$, a constant. Therefore, $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$. We denote $[\lambda]'$ and $[-\lambda]'$ as the corresponding eigenspaces associated with h' corresponding to the non-zero eigen values λ and $-\lambda$ respectively. We have the following lemmas.

Lemma 2.1. ([5], Proposition 4.1) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k, \mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1, \mu = -2$ and $Spec(h') = \{0, \lambda, -\lambda\}$ with 0 as simple eigen value and $\lambda = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\lambda]'$ and $[\xi] \oplus [-\lambda]'$ are integrable with totally geodesic leaves. The distributions $[\lambda]'$ and $[-\lambda]'$ are integrable with totally umbilical leaves.

Lemma 2.2. ([5], Lemma 4.1) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with $h' \neq 0$ and ξ belongs to the $(k, -2)'$-nullity distribution. Then for every $X, Y \in T_pM$,

$$(\nabla_X h')Y = -g(h'X + h'^2X, Y)\xi - \eta(Y)(h'X + h'^2X). \tag{2.12}$$

According to Takahashi [12] and De et al. [4], we have the following definitions:

Definition 2.1. An almost Kenmotsu manifold is said to be ϕ-symmetric if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = 0, \tag{2.13}$$

for all vector fields $W, X, Y, Z \in T_pM^{2n+1}$. In addition, if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-symmetric

Definition 2.2. An almost Kenmotsu manifold is said to be ϕ-recurrent if it satisfies

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z \tag{2.14}$$

for any vector fields $W, X, Y, Z \in T_pM^{2n+1}$. In relation (2.14) if the vector fields W, X, Y, Z are orthogonal to ξ, then the manifold is called locally ϕ-recurrent.

3. M-projectively semisymmetric almost Kenmotsu manifold with ξ belongs to $(k, \mu)'$−nullity distribution

Let M^{2n+1} be M-projectively semisymmetric almost Kenmotsu manifold with ξ belongs to $(k, \mu)'$-nullity distribution. Then we have

$$(R(X, Y) \cdot W^*)(U, V)W = 0, \tag{3.1}$$

for all vector fields X, Y, U, V, W. This implies

$$\begin{aligned} &R(X, Y)W^*(U, V)W - W^*(R(X, Y)U, V)W \\ &\quad - W^*(U, R(X, Y)V)W - W^*(U, V)R(X, Y)W = 0. \end{aligned} \tag{3.2}$$

Putting $X = U = \xi$ in(3.2), we get

$$\begin{aligned} &R(\xi, Y)W^*(\xi, V)W - W^*(R(\xi, Y)\xi, V)W \\ &\quad - W^*(\xi, R(\xi, Y)V)W - W^*(\xi, V)R(\xi, Y)W = 0. \end{aligned} \tag{3.3}$$

Using (2.10) and lemma (2.1) in (1.5), we get

$$\begin{aligned} R(\xi, Y)W^*(\xi, V)W = k\{&(k + 2nk + \frac{1}{2})g(V, W)\eta(Y)\xi - \frac{3}{2}g(h'V, W)\eta(Y)\xi \\ &- \frac{(k+1)}{2}g(Y, V)\eta(W)\xi + \frac{3}{2}g(Y, h'V)\eta(W)\xi \\ &- (k + 2nk + \frac{1}{2})g(V, W)Y + \frac{3}{2}g(h'V, W)Y \\ &- 2\{-\frac{(k+1)}{2}g(h'Y, V)\eta(W)\xi + \frac{3}{2}g(h'Y, h'V)\eta(W)\xi \\ &+ \frac{(k+1)}{2}\eta(V)\eta(W)Y\} - (k + 2nk + \frac{1}{2})g(V, W)h'Y \\ &+ \frac{3}{2}g(h'V, W)h'Y + \frac{(k+1)}{2}\eta(V)\eta(W)h'Y\}. \end{aligned} \tag{3.4}$$

Making use of (2.10) and (1.5), we obtain the following:

$$\begin{aligned} W^*(R(\xi, Y)\xi, V)W = k\{&(k + 2nk + \frac{1}{2})g(V, W)\eta(Y)\xi - \frac{3}{2}g(h'V, W)\eta(Y)\xi \\ &- \frac{(k+1)}{2}\eta(W)\eta(Y)V + \frac{3}{2}\eta(Y)\eta(W)h'V \\ &- W^*(Y, V)W\} + 2W^*(h'Y, V)W, \end{aligned} \tag{3.5}$$

$$\begin{aligned} W^*(\xi, R(\xi, Y)V)W = k\{&(\frac{k}{2} + 2nk)g(Y, V)\eta(W)\xi - (k + 2nk + \frac{1}{2})g(Y, W) \\ &\eta(V)\xi + \frac{3}{2}g(h'Y, W)\eta(V)\xi + \frac{(k+1)}{2}\eta(V)\eta(W)Y \\ &- \frac{3}{2}\eta(V)\eta(W)h'Y - (4n+1)g(h'Y, V)\eta(W)\xi\} \\ &+ 2(k + 2nk + \frac{1}{2})g(h'Y, W)\eta(V)\xi - (k+1) \\ &[3g(Y, W)\eta(V)\xi - 6\eta(Y)\eta(V)\eta(W)\xi \\ &+ \eta(V)\eta(W)h'Y + 3\eta(V)\eta(W)Y] \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}W^*(\xi,V)R(\xi,Y)W = k\{&(\frac{k}{2}+2nk)g(Y,W)\eta(V)\xi+\frac{3}{2}g(Y,W)h'V\\&-(k+2nk+\frac{1}{2})g(Y,V)\eta(W)\xi+\frac{3}{2}g(h'V,Y)\eta(W)\xi\\&+\frac{(k+1)}{2}\eta(Y)\eta(W)V-\frac{3}{2}\eta(W)\eta(Y)h'V\}\\&-2\{(\frac{k}{2}+2nk)g(h'Y,W)\eta(V)\xi+\frac{3}{2}g(h'Y,W)h'V\\&-(k+2nk+\frac{1}{2})g(h'Y,V)\eta(W)\xi+\frac{3}{2}g(h'V,h'Y)\eta(W)\xi\\&+\frac{(k+1)}{2}\eta(Y)\eta(W)V\}. \qquad (3.7)\end{aligned}$$

Using (3.4)-(3.7) in (3.3), we obtain

$$\begin{aligned}&kW^*(Y,V)W+(k+1)(\frac{k}{2}+3)g(Y,W)\eta(V)\xi-k(k+2nk+\frac{1}{2})g(V,W)Y\\&+3(k+1)\eta(V)\eta(W)Y-6(k+1)\eta(Y)\eta(V)\eta(W)\xi+\frac{3k}{2}g(h'V,W)Y\\&-(\frac{5k}{2}+1)g(h'Y,W)\eta(V)\xi+2(k+2nk+\frac{k}{2})g(V,W)h'Y\\&-\frac{3k}{2}g(Y,W)h'V+\frac{3k}{2}\eta(V)\eta(W)h'Y-3g(h'V,W)h'Y\\&+3g(h'Y,W)h'V-2W^*(h'Y,V)W=0. \qquad (3.8)\end{aligned}$$

Letting $Y,W\in[\lambda]'$ and $V\in[-\lambda]'$, we obtain from (3.8) that

$$W^*(Y,V)W=(k+1)g(Y,W)V \quad and \quad W^*(h'Y,V)W=\lambda(k+1)g(Y,W)V. \qquad (3.9)$$

In view of (3.9), (3.8) becomes

$$[\frac{3k\lambda}{2}-3\lambda^2+(k+1)(k-2\lambda)]g(Y,W)V=0. \qquad (3.10)$$

Using the relation $\lambda^2=-(k+1)$ in (3.10), we get

$$\lambda(\lambda-1)(2\lambda^2+3\lambda-1)=0. \qquad (3.11)$$

If $\lambda=0$, then $k=-1$ and hence from (2.8) we have $h'=0$, which is a contradiction to our assumption that $h'\neq 0$. Then from (3.11) without loss of generality we may take $\lambda=1$ and hence $k=-2$. Hence from propositions 4.2 and 4.3 of [5], we have the following:

Theorem 3.1. The M-projective semisymmetric almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)(n>1)$ with ξ belonging to the $(k,\mu)'$-nullity distribution and

$h' \neq 0$ is locally isometric to the Riemannian product of an $(n+1)$- dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.

4. $M-$projectively Ricci-semisymmetric almost Kenmotsu manifold with ξ belongs to $(k,\mu)'-$nullity distribution

Definition 4.1. An almost Kenmotsu manifold is M-projectively Ricci- semisymmetric if

$$(W^*(X,Y)\cdot S)(U,V)=0, \tag{4.1}$$

for all vector fields X,Y,U,V.

From (4.1), we have

$$S(W^*(X,Y)U,V)+S(U,W^*(X,Y)V)=0. \tag{4.2}$$

Setting $X=U=\xi$ in (4.2), we get

$$S(W^*(\xi,Y)\xi,V)+S(\xi,W^*(\xi,Y)V)=0. \tag{4.3}$$

Using (1.5) and (2.11) in (4.3), we obtain

$$\begin{aligned}&-(\frac{k+1}{2})S(Y,V)+\frac{3}{2}S(h'Y,V)+2nk(k+2nk+\frac{1}{2})g(Y,V)\\&-3nkg(h'Y,V)+nk^2(4n+1)\eta(Y)\eta(V)=0.\end{aligned} \tag{4.4}$$

Replacing Y by $h'Y$ in (4.4), we get

$$(k+1)\{-3S(Y,V)-S(h'Y,V)+6nkg(Y,V)\}+4nk(k+2nk+\frac{1}{2})g(h'Y,V)=0. \tag{4.5}$$

Letting $Y,V\in[\lambda]'$ in (4.5) gives that

$$S(Y,V)=\frac{nk}{\lambda+3}[3+\frac{2\lambda(4nk+2k+1)}{k+1}]g(Y,V). \tag{4.6}$$

Using lemma (2.1), from (4.6) we can state the following:

Theorem 4.1. An M-projectively Ricci-semisymmetric almost Kenmotsu manifold M^{2n+1} with ξ belongs to $(k,\mu)'$-nullity distribution is an Einstein manifold provided $\lambda\neq-3$.

5. $M-$projectively semisymmetric almost Kenmotsu manifold with ξ belongs to $(k,\mu)-$nullity distribution

For $\xi\in(k,\mu)-$nullity distribution, we have from (1.3) that

$$R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]+\mu[g(Y,Z)hX-g(X,Z)hY]. \tag{5.1}$$

Using theorem 4.1 of [5], it follows from (5.1) that

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X], \tag{5.2}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{5.3}$$

$$S(X,\xi) = -2n\eta(X). \tag{5.4}$$

Applying (5.3) and (5.4) in (1.5), we get

$$W^*(\xi,Y)Z = -g(Y,Z)\xi + \eta(Z)Y - \frac{1}{4n}[S(Y,Z)\xi + 2n\eta(Z)Y + g(Y,Z)Q\xi - \eta(Z)QY].$$

Suppose

$$(R(X,Y)\cdot W^*)(U,V)W = 0 \tag{5.5}$$

for all vector fields X, Y, U, V, W. Then we have

$$\begin{aligned} &R(X,Y)W^*(U,V)W - W^*(R(X,Y)U,V)W \\ &- W^*(U,R(X,Y)V)W - W^*(U,V)R(X,Y)W = 0. \end{aligned} \tag{5.6}$$

Putting $X = U = \xi$ in(5.6), we get

$$\begin{aligned} &R(\xi,Y)W^*(\xi,V)W - W^*(R(\xi,Y)\xi,V)W \\ &- W^*(\xi,R(\xi,Y)V)W - W^*(\xi,V)R(\xi,Y)W = 0. \end{aligned} \tag{5.7}$$

Using (5.3), (5.4) and (5.5), we get

$$\begin{aligned} R(\xi,Y)W^*(\xi,V)W = &\frac{1}{2}g(V,W)\eta(Y)\xi - \frac{1}{2}g(Y,V)\eta(W)\xi - \frac{1}{4n}S(V,W)\eta(Y)\xi \\ &- \frac{1}{4n}S(Y,V)\eta(W)\xi - \frac{1}{2}g(V,W)Y - \frac{1}{4n}S(V,W)Y, \end{aligned} \tag{5.8}$$

$$\begin{aligned} W^*(R(\xi,Y)\xi,V)W = &W^*(Y,V)W - \frac{1}{2}g(V,W)\eta(Y)\xi + \frac{1}{2}\eta(W)\eta(Y)V \\ &- \frac{1}{4n}[S(V,W)\eta(Y)\xi - \eta(Y)\eta(W)QV], \end{aligned} \tag{5.9}$$

$$\begin{aligned} W^*(\xi,R(\xi,Y)V)W = &-g(Y,W)\eta(V)\xi + \eta(V)\eta(W)Y - \frac{1}{4n}[S(Y,W)\eta(V)\xi \\ &+ 2n\eta(V)\eta(W)Y + g(Y,W)\eta(V)Q\xi - \eta(W)\eta(V)QY] \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} W^*(\xi,V)R(\xi,Y)W = &-\frac{1}{2}g(Y,W)V - \frac{1}{4n}g(Y,W)QV - \frac{1}{2}g(Y,V)\eta(W)\xi \\ &+ \frac{1}{2}\eta(Y)\eta(W)V - \frac{1}{4n}S(Y,V)\eta(W)\xi + \frac{1}{4n}\eta(W)\eta(Y)QV. \end{aligned} \tag{5.11}$$

Using (5.8)-(5.11) in (5.7), we obtain

$$\begin{aligned} W^*(Y,V)W &= g(V,W)\eta(Y)\xi + \frac{1}{2n}S(V,W)\eta(Y)\xi - \frac{1}{2}g(V,W)Y \\ &- \eta(Y)\eta(W)V - \frac{1}{2n}\eta(Y)\eta(W)QV + \frac{1}{2}g(Y,W)\eta(V)\xi \\ &- \frac{1}{2}\eta(V)\eta(W)Y + \frac{1}{2}g(Y,W)V + \frac{1}{4n}[S(Y,W)\eta(V)\xi \\ &- S(V,W)Y - \eta(V)\eta(W)QY + g(Y,W)QV]. \end{aligned} \tag{5.12}$$

By the influence of (1.5), one can get from (5.12) that

$$\begin{aligned} R(Y,V)W &= g(V,W)\eta(Y)\xi + \frac{1}{2n}S(V,W)\eta(Y)\xi - \frac{1}{2}g(V,W)Y \\ &- \eta(Y)\eta(W)V - \frac{1}{2n}\eta(Y)\eta(W)QV + \frac{1}{2}g(Y,W)\eta(V)\xi \\ &- \frac{1}{2}\eta(V)\eta(W)Y + \frac{1}{2}g(Y,W)V + \frac{1}{4n}[S(Y,W)\eta(V)\xi \\ &- S(Y,W)V - \eta(V)\eta(W)QY + g(V,W)QY]. \end{aligned} \tag{5.13}$$

Contracting (5.13) over Y, we get

$$S(V,W) = \frac{(r+4n-4n^2)}{4n-1}g(V,W) - \frac{(r+2n+4n^2)}{4n-1}\eta(V)\eta(W). \tag{5.14}$$

Again contracting (5.14) over V and W, we get $r = -2n - 4n^2$ and substituting this in (5.14), we get

$$S(V,W) = -2ng(V,W). \tag{5.15}$$

On account of (5.15), it follows from (5.13) that

$$R(Y,V)W = -[g(V,W)Y - g(Y,W)V]. \tag{5.16}$$

Hence from (5.16), we can state the following:

Theorem 5.1. An M-projectively semisymmetric almost Kenmotsu manifold M^{2n+1} with ξ belongs to (k,μ)-nullity distribution is an Einstein manifold and moreover the manifold is of constant curvature -1.

6. $M-$projectively Ricci-semisymmetric almost Kenmotsu manifold with ξ belongs to $(k,\mu)-$nullity distribution

Definition 6.1. An almost Kenmotsu manifold is M-projectively Ricci- semisymmetric if

$$(W^*(X,Y)\cdot S)(U,V) = 0, \tag{6.1}$$

for all vector fields X, Y, U, V.

From (4.3), we have

$$S(W^*(\xi, Y)\xi, V) + S(\xi, W^*(\xi, Y)V) = 0. \tag{6.2}$$

Using (1.5) and theorem 4.1 of [5] in (6.2), we obtain

$$S(Y, V) = -ng(Y, V) - \frac{1}{4n} S(QY, V) \tag{6.3}$$

Taking $V = \xi$ in (6.3), we get

$$QY = -2nY \tag{6.4}$$

Therefore (6.3) becomes

$$S(Y, V) = -2ng(Y, V). \tag{6.5}$$

This implies that the manifold is Einstien manifold.
Conversely let the manifold be Einstien manifold of the form (6.5). Then clearly $W^* \cdot S = 0$. Hence we can state the following:

Theorem 6.1. An almost Kenmotsu manifold M^{2n+1} with ξ belongs to (k, μ)-nullity distribution is an Einstein manifold if and only if it is M-projectively Ricci-semisymmetric.

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