

Some Properties of Curvature Tensors on Quarter-Symmetric Metric Connection in a P-Sasakian Manifold

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Abstract

The object of the present paper is to study properties of concircular and conharmonic curvature tensor of a quarter symmetric metric connection in a P-Sasakian manifold.

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1. Introduction

An n -dimensional differentiable manifold M is called an almost para-contact manifold if it admits an almost para-contact structure (F, ξ, η) consisting of a $(1, 1)$ tensor field F , a vector field ξ and a 1-form η satisfying

$$\bar{X} = X - \eta(X) \xi, \quad (1.1)$$

$$\bar{X} = F(X) \quad (1.2)$$

$$\eta(\xi) = 1. \quad (1.3)$$

Let g be the compatible Riemannian metric with (F, ξ, η) that is

$$g(FX, FY) = g(X, Y) - \eta(X) \eta(Y) \quad (1.4)$$

or equivalently

$$g(X, FY) = g(FX, Y) \quad (1.5)$$

and

$$g(X, \xi) = \eta(X) \quad \text{for all } X, Y \in TM \quad (1.6)$$

Then M becomes almost para-contact Riemannian manifold equipped with an almost para-contact Riemannian structure (F, ξ, η, γ) . An almost para-contact Riemannian manifold is called a P-Sasakian manifold if it satisfies

$$(\nabla_X F) Y = -g(X, Y) \xi - \eta(Y) X + 2\eta(X) \eta(Y) \xi, \quad X, Y \in TM \quad (1.7)$$

where ∇ denote the covariant differentiation with respect to g . It follows that

$$(\nabla_X \xi) = \bar{X}, \quad (1.8)$$

$$(\nabla_X \eta) = (\nabla_Y \eta) X = g(X, \bar{Y}), \quad X \in TM. \quad (1.9)$$

In an n -dimensional P-Sasakian manifold M , the curvature tensor R , the Ricci tensor S and the Ricci operator Q , satisfy

$$R(X, Y) \xi = \eta(X) Y - \eta(Y) X \quad (1.10)$$

$$R(\xi, X) Y = \eta(Y) X - g(X, Y) \xi \quad (1.11)$$

$$R(\xi, X) \xi = X - \eta(X) \xi \quad (1.12)$$

$$S(X, \xi) = -(n-1) \eta(X) \quad (1.13)$$

$$Q\xi = -(n-1) \xi \quad (1.14)$$

$$\eta(R(X, Y) U) = g(X, U) \eta(Y) - g(Y, U) \eta(X) \quad (1.15)$$

$$\eta(R(X, Y) \xi) = 0 \quad (1.16)$$

$$\eta(R(\xi, X) Y) = \eta(X) \eta(Y) - g(X, Y). \quad (1.17)$$

An almost para-contact Riemannian manifold M is said to be η -Einstein [2], if the Ricci operator Q satisfying

$$Q(X) = aX + b \eta(X) \xi$$

where a and b are smooth function on the manifold. In particular if $b = 0$, the manifold is an Einstein manifold.

Let (M, g) be an n -dimensional Riemannian manifold. Then the concircular curvature tensor V and the conharmonic curvature tensor L are given by

$$V(X, Y, Z) = R(X, Y, Z) - \frac{r}{n(n-1)} [g(Y, Z)X - g(X, Z)Y] \quad (1.18)$$

$$L(X, Y, Z) = R(X, Y, Z) - \frac{1}{(n-2)} [\text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \quad (1.19)$$

for all $X, Y \in TM$ respectively, where r is the scalar curvature of M .

A linear connection $\tilde{\nabla}$ in a Riemannian manifold M is said to be a quarter symmetric connection if its torsion tensor T satisfies

$$T(X, Y) = \eta(Y)F(X) - \eta(X)F(Y) \quad (1.20)$$

where η is a 1-form and F is a $(1, 1)$ tensor field [4].

A linear connection $\tilde{\nabla}$ satisfying (1.20) and (1.21) is called a quarter symmetric metric connection [3].

2. Curvature Tensor

We consider a linear connection and be a Riemannian connection such that [6]

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y) \quad (2.1)$$

where U is a tensor of type $(1, 2)$, and

$$T(X, Y) = \eta(Y)\bar{X} - \eta(X)\bar{Y} = U(X, Y) - U(Y, X). \quad (2.2)$$

If a connection $\tilde{\nabla}$ is metric connection, i.e.

$$(\tilde{\nabla}_X g)(Y, Z) = 0 \quad (2.3)$$

holds.

From (2.2), we have

$$'U(X, Y, Z) + 'U(X, Z, Y) = 0 \quad (2.4)$$

where $'U(X, Y, Z) = g(U(X, Y), Z)$.

Since

$$\begin{aligned}(\tilde{\nabla}_X g)(Y, Z) = 0 &\Rightarrow g(\tilde{\nabla}_X Y, Z) + g(Y, \tilde{\nabla}_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ &\Rightarrow g(U(X, Y), Z) + g(Y, U(X, Z)) = 0 \\ &\Rightarrow 'U(X, Y, Z) + 'U(X, Z, Y) = 0\end{aligned}$$

and also

$$U(X, Y) = \frac{1}{2} [T(X, Y) + T'(X, Y) + T'(Y, X)] \quad (2.5)$$

where $g(T'(Y, X), Z) = g(T(Z, X), Y)$ see [3] (2.6)

Assume that the torsion tensor $T(X, Y)$ of the linear connection is of the form

$$T(X, Y) = \eta(Y) \bar{X} - \eta(X) \bar{Y}. \quad (2.7)$$

From (2.6) and (2.7), we have

$$T(X, Y) = \eta(X) Y - 'F(X, Y) \quad (2.8)$$

where $'F(X, Y) = g(\bar{X}, Y)$, η is a 1-form and ξ is the associated vector field.

From (2.5), (2.7) and (2.8), we get

$$U(X, Y) = \eta(Y) \bar{X} - 'F(X, Y) \xi. \quad (2.9)$$

From (2.1) and (2.9), we get

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y) \bar{X} - 'F(X, Y) \xi. \quad (2.10)$$

Hence a quarter symmetric metric connection $\tilde{\nabla}$ in a P-Sasakian manifold is given by (2.10).

If R and \tilde{R} be the curvature tensors of the connection ∇ and $\tilde{\nabla}$ respectively. Then we have

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \quad (2.11)$$

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z. \quad (2.12)$$

Using (2.10) and (2.11) in (2.12), we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + 3 'F(X, Z)Y - 3 'F(Y, Z)X + [(\nabla_X F)(Y) \\ &\quad - (\nabla_Y F)(X)] \eta(Z) - [(\nabla_X 'F)(Y, Z) - (\nabla_Y 'F)(X, Z)] \xi. \end{aligned} \quad (2.13)$$

From (1.7) and (2.13), we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + 3 'F(X, Z)\bar{Y} - 3 'F(Y, Z)\bar{X} + [\eta(X)Y \\ &\quad - \eta(Y)X] \eta(Z) - [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)] \xi. \end{aligned} \quad (2.14)$$

From (2.14), we have

$$\begin{aligned} \tilde{R}(X, Y, Z, U) &= 'R(X, Y, Z, U) + 3 'F(X, Z)'F(Y, U) - 3 'F(Y, Z)'F(X, U) \\ &\quad + \eta(X)\eta(Z)g(Y, U) - \eta(Y)\eta(Z)g(X, U) \\ &\quad - [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\eta(U). \end{aligned} \quad (2.15)$$

A relation between the curvature tensor of M with respect to the quarter-symmetric connection $\tilde{\nabla}$ and the Riemannian connection, ∇ is given by the equation (2.15).

Putting $X = U = e_i$ in (2.15) where $\{e_i\}$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over i , $1 \leq i \leq n$, we get

$$\begin{aligned} \tilde{S}(Y, Z) &= S(Y, Z) + \eta(Y)\eta(Z) - n\eta(Y)\eta(Z) - g(Y, Z) + \eta(Y)\eta(Z) \\ \tilde{S}(Y, Z) &= S(Y, Z) - (n-2)\eta(Y)\eta(Z) - g(Y, Z) \end{aligned} \quad (2.16)$$

contracting (2.16) with respect to Z , we get

$$\tilde{r} = r - 2(n-1) \quad (2.17)$$

where \tilde{r} and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇ respectively.

From (2.14) and (2.15), we have

$$\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0$$

$$\tilde{R}(X, Y, Z, U) + \tilde{R}(X, Y, U, Z) = 0$$

and $\tilde{R}(X, Y, Z, U) - \tilde{R}(Z, U, X, Y) = 0$

putting $U = \xi$ in (2.15), we have

$$\tilde{R}(X, Y, Z, \xi) = 2 \tilde{R}(X, Y, Z, \xi).$$

Putting $Y = Z = e_i$ in (2.15) and taking summation over i , we have

$$\tilde{S}(X, \xi) = 2S(X, \xi).$$

Thus we can state the following theorem :

Theorem 1. For a P-Sasakian manifold M with quarter symmetric metric connection $\tilde{\nabla}$, we have

(a) $\tilde{R}(X, Y)Z + \tilde{R}(Y, Z)X + \tilde{R}(Z, X)Y = 0$

(b) $\tilde{R}(X, Y, Z, U) + \tilde{R}(X, Y, U, Z) = 0$

(c) $\tilde{R}(X, Y, Z, U) - \tilde{R}(Z, U, X, Y) = 0$

(d) $\tilde{R}(X, Y, Z, \xi) = 2 \tilde{R}(X, Y, Z, \xi)$

(e) $\tilde{S}(X, \xi) = 2S(X, \xi).$

The concircular curvature tensor \tilde{V} of type $(0, 4)$ of M^n with respect to Riemannian connection is given by

$$\tilde{V}(X, Y, Z, U) = \tilde{R}(X, Y, Z, U) - \frac{1}{n(n-1)} [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \quad (2.19)$$

Analogous to this definition, we define concircular curvature tensor of M^n with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$\tilde{V}(X, Y, Z, U) = \tilde{R}(X, Y, Z, U) - \frac{\tilde{r}}{n(n-1)} [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]. \quad (2.20)$$

From (2.15), (2.17), (2.19) and (2.20), we have

$$\begin{aligned} \tilde{V}(X, Y, Z, U) = & 'V(X, Y, Z, U) + 3 'F(X, Z) 'F(Y, U) - 3 'F(Y, Z) 'F(X, U) \\ & + \eta(X) \eta(Z) g(Y, U) - \eta(Y) \eta(Z) g(X, U) - [\eta(X) g(Y, Z) \\ & - \eta(Y) g(X, Z)] \eta(U) + \frac{2}{n} [g(Y, Z) g(X, U) - g(X, Z) g(Y, U)]. \end{aligned} \quad (2.21)$$

From (2.21), we get

$$\tilde{V}(X, Y, Z, U) + \tilde{V}(Y, Z, X, U) + \tilde{V}(Z, X, Y, U) = 0.$$

Putting $U = \xi$ in (2.21), we have

$$\tilde{V}(X, Y, Z, \xi) = \frac{r-2(n-1)}{n(n-1)} [g(X, Z) \eta(Y) - g(Y, Z) \eta(X)].$$

Hence we can state the following theorem :

Theorem 2. In a P-Sasakian manifold the concircular curvature \tilde{V} of a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying

- (i)
$$\begin{aligned} \tilde{V}(X, Y, Z, U) = & 'V(X, Y, Z, U) + 3 'F(X, Z) 'F(Y, U) - 3 'F(Y, Z) 'F(X, U) \\ & + \eta(X) \eta(Z) g(Y, U) - \eta(Y) \eta(Z) g(X, U) - [\eta(X) g(Y, Z) \\ & - \eta(Y) g(X, Z)] \eta(U) + \frac{2}{n} [g(Y, Z) g(X, U) - g(X, Z) g(Y, U)] \end{aligned}$$
- (ii)
$$\tilde{V}(X, Y, Z, U) + \tilde{V}(Y, Z, X, U) + \tilde{V}(Z, X, Y, U) = 0$$
- (iii)
$$\tilde{V}(X, Y, Z, \xi) = \frac{r-2(n-1)}{n(n-1)} [g(X, Z) \eta(Y) - g(Y, Z) \eta(X)].$$

The conharmonic curvature tensor 'L of type (0, 4) of M^n with respect to the Riemannian connection is given by

$$\begin{aligned} 'L(X, Y, Z, U) = & 'R(X, Y, Z, U) - \frac{1}{n-2} [S(Y, Z) g(X, U) - S(X, Z) g(Y, U) \\ & + g(Y, Z) S(X, U) - g(X, Z) S(Y, U)]. \end{aligned} \quad (2.22)$$

Analogous to this definition, we define conharmonic curvature tensor of M^n with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ by

$$\begin{aligned} \tilde{L}(X, Y, Z, U) = & \tilde{R}(X, Y, Z, U) - \frac{1}{n-2} [\tilde{S}(Y, Z)g(X, U) - \tilde{S}(X, Z)g(Y, U) \\ & + g(Y, Z)\tilde{S}(X, U) - g(X, Z)\tilde{S}(Y, U)]. \end{aligned} \quad (2.23)$$

From (2.15), (2.16), (2.17), (2.22) and (2.23), we have

$$\begin{aligned} \tilde{L}(X, Y, Z, U) = & 'L(X, Y, Z, U) + 3 'F(X, Z) 'F(Y, U) - 3 'F(Y, Z) 'F(X, U) \\ & - \frac{2}{n-2} [g(X, Z)g(Y, U) - g(Y, Z)g(X, U)]. \end{aligned} \quad (2.24)$$

From (2.24), we get

$$\tilde{L}(X, Y, Z, U) + \tilde{L}(Y, Z, X, U) + \tilde{L}(Z, X, Y, U) = 0.$$

Putting $U = \xi$ in (2.24), we have

$$\tilde{L}(X, Y, Z, \xi) = \frac{1}{n-2} [\{3g(Y, Z) - S(Y, Z)\} \eta(X) - \{3g(X, Z) - S(X, Z)\} \eta(Y)].$$

Hence we can state the following theorem :

Theorem 3. In a P-Sasakian manifold the conharmonic curvature \tilde{L} of a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying

- (i) $\tilde{L}(X, Y, Z, U) = 'L(X, Y, Z, U) + 3 'F(X, Z) 'F(Y, U) - 3 'F(Y, Z) 'F(X, U)$
 $\quad - \frac{2}{n-2} [g(X, Z)g(Y, U) - g(Y, Z)g(X, U)]$
- (ii) $\tilde{L}(X, Y, Z, U) + \tilde{L}(Y, Z, X, U) + \tilde{L}(Z, X, Y, U) = 0.$
- (iii) $\tilde{L}(X, Y, Z, \xi) = \frac{1}{n-2} [\{3g(Y, Z) - S(Y, Z)\} \eta(X)$
 $\quad - \{3g(X, Z) - S(X, Z)\} \eta(Y)].$

3. Ricci-parallel P-Sasakian manifold with respect to quarter-symmetric metric connection

A p-Sasakian manifold is Ricci-parallel if [5]

$$S(Y, Z) = -(n-1)g(Y, Z). \quad (3.1)$$

Using (3.1) in (2.16), we get

$$\tilde{S}(Y, Z) = -ng(Y, Z) - (n-2)\eta(Y)\eta(Z). \quad (3.2)$$

Hence we can state the following theorem :

Theorem 4. In a Ricci-parallel p-Sasakian manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$, we have

$$\tilde{S}(Y, Z) = -ng(Y, Z) - (n-2)\eta(Y)\eta(Z).$$

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