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On a Special (α, β)-Metric and its Hypersurface

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1. Introduction

A Finsler metric L(x, y) is called an (α, β) metric if it is positively homogeneous function of degree one in Riemannian metric $\alpha = (a_{ij}(x)\, y^i y^j)^{1/2}$ and 1-form $\beta = b_i(x)\, y^i$, [1], [2]. Some of the well known (α, β)-metrics are Randers metric, Kropina metric, Generalized Kropina metric and motsumoto metric. In 1995, Hong-Suh Park and Eun Seo Choi [4] introduced a special (α, β)-metric given by

$$L^2 = c_1 \alpha^2 + 2c_2 \alpha \beta + c_3 \beta^2 \tag{1.1}$$

where c_1, c_2 and c_3 are constants.

In the present paper, I introduce another special (α, β)-metric given by

$$L^3 = c_1 \alpha^3 + 3c_2 \alpha^2 \beta + 3c_3 \alpha\beta^2 + c_4 b^3 \tag{1.2}$$

In 1995, M. Matsumoto [3] had discussed the properties of special hypersurface of Rander space with $b_i(x) = (\partial_i b)$ being the gradient of a scalar function b(x). He had considered a hypersurface which is given by b(x) = constant.

In this paper I have considered the hypersurface given by the equation b(x) = constant of the Finsler space with special (α, β)-metric given by (1.2).

2. The Finsler Space with Metric (1.2)

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space with (α, β) metric given by equation (1.2), where $\alpha = (a_{ij}(x)\, y^i y^j)^{1/2}$ is a Riemannian metric in M^n and $\beta = b_i(x)\, y^i$ is a 1-form in M^n. The derivatives of L(α, β) with respect to α and β are given by

$$L_\alpha = L^{-2}(c_1\,\alpha^2 + 2c_2\,\alpha\,\beta + c_3\,\beta^2) \tag{2.1}$$

$$L_\beta = L^{-2}(c_2\,\alpha^2 + 2c_3\,\alpha\,\beta + c_4\,\beta^2) \tag{2.2}$$

$$L_{\alpha\alpha} = 2L^{-2}(c_1\,\alpha + c_2\,\beta - LL_\alpha^2) \tag{2.3}$$

$$L_{\beta\beta} = 2L^{-2}(c_3\,\alpha + c_4\,\beta - LL_\beta^2) \tag{2.4}$$

$$L_{\alpha\beta} = 2L^{-2}(c_2\,\alpha + c_3\,\beta - LL_\alpha L_\beta) \tag{2.5}$$

where $L_\alpha = \partial L/\partial\alpha$, $L_\beta = \partial L/\partial\beta$, $L_{\alpha\alpha} = \partial L_\alpha/\partial\alpha$ and $L_{\beta\beta} = \partial L_\beta/\partial\beta$.

The normalized element of support $l_i = \partial L/\partial y^i$, is given by

$$l_i = \alpha^{-1} L_\alpha\, y_i + L_\beta\, b_i \tag{2.6}$$

where $y_i = a_{ij}\, y^j$. The angular metric tensor $h_{ij} = L = (\partial^2 L/\partial y_i\,\partial y_j)$ is given by

$$h_{ij} = p\,a_{ij} + q_0\,b_i\,b_j + q_1\,(b_i\,y_j + b_j\,y_i) + q_2\,y_i\,y_j, \tag{2.7}$$

where

$$p = \alpha^{-1}LL_\alpha = \alpha^{-1}L^{-1}(c_1\,\alpha^2 + 2c_2\,\alpha\,\beta + c_3\,\beta^2)$$

$$q_0 = LL_{\beta\beta} = 2L^{-1}(c_3\,\alpha + c_4\,\beta - LL_\beta^2) \tag{2.8}$$

$$q_1 = \alpha^{-1}LL_{\alpha\beta} = 2\alpha^{-1}L^{-1}(c_2\,\alpha + c_3\,\beta - LL_\alpha L_\beta)$$

and $$q_2 = \alpha^{-2}L(L_{\alpha\alpha} - \alpha^{-1}L_\alpha) = \alpha^{-3}L^{-1}(c_1\,\alpha^2 - 2\,\alpha\,LL_\alpha^2 - c_3\,\beta^2)$$

The fundamental tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\,\partial y^j}$ is given by

$$g_{ij} = p\,a_{ij} + p_0\,b_i\,b_j + p_1\,(b_i\,y_j + b_j\,y_i) + p_2\,y_i\,y_j, \tag{2.9}$$

where

$$p_0 = q_0 + L_\beta^2 = 2L^{-1}\{c_3\,\alpha + c_4\,\beta - \tfrac{1}{2}LL_\beta^2\}$$

$$p_1 = q_1 + L^{-1}pL_\beta = 2\alpha^{-1}L^{-1}(c_2\,\alpha + c_3\,\beta - \tfrac{1}{2}LL_\alpha L_\beta) \tag{2.10}$$

and $$p_2 = q_2 + \alpha^{-2}L_\alpha^2 = \alpha^{-3}L^{-1}(c_1\,\alpha^2 - \alpha\,LL_\alpha^2 - c_3\,\beta^2).$$

The reciprocal tensor g^{ij} of g_{ij} is given by

$$g^{ij} = p^{-1} a^{ij} - s_0 b^i b^j - s_1 (b^i y^j + b^j y^i) - s_2 y^i y^j, \tag{2.11}$$

where

$$b^i = a^{ij} b_j,$$

$$J = p\,(p + p_0 b^2 - 2p_1\beta + p_2 \alpha^2) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2)$$

$$s_0 = \frac{1}{JP} \{pp_0 + (p_0 p_2 - p_1^2)\,\alpha^2\} \tag{2.12}$$

$$s_1 = \frac{1}{JP} \{pp_1 + (p_0 p_2 - p_1^2)\,\beta\}$$

$$s_2 = \frac{1}{JP} \{pp_2 + (p_0 p_2 - p_1^2)\, b^2\}, \qquad b^2 = a_{ij} b^i b^j.$$

3. The Hypersurfaces $F^{n-1}(c)$

In this section, I consider a special (α, β) metric (1.2) with a gradient $b_i(x) = \partial_i b$ for a scalar function b(x) and also consider a hypersurface F^{n-1} given by the equation b(x) = c (constant).

Since the parametric equation of $F^{n-1}(c)$ is $x^i = x^i(u^\alpha)$, hence $(\partial/\partial u^\alpha)\, b(x(u)) = 0 = b_i(x)\, X^i_\alpha$, where $b_i(x)$ are considered as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$, we have

$$b_i X^i_\alpha = 0, \qquad b_i y^i = 0. \tag{3.1}$$

In general, the induced metric $\underline{L}(u, v)$ from the metric (1.2) is given by

$$\underline{L}^3 = c_1 \{a_{\alpha\beta}(u)\, v^\alpha v^\beta\}^{3/2}, \quad \text{where } a_{\alpha\beta} = a_{ij}(x(u))\, X^i_\alpha X^j_\beta \tag{3.2}$$

which is a Riemannian metric at the point of $F^{n-1}(c)$. From (2.8), (2.10) and (2.12), we have

$$p = c_1^{2/3}, \qquad q_0 = 2\, c_1^{-4/3}(c_1 c_3 - c_2^2), \qquad q_1 = 0, \qquad q_2 = -\alpha^{-2} c_1^{2/3},$$

$$p_0 = c_1^{-4/3}(2\, c_1 c_3 - c_2^2), \qquad p_1 = \alpha^{-1} c_1^{-1/3} c_2, \qquad p_2 = 0,$$

$$J = c_1^{4/3} + 2\, c_1^{-2/3}(c_1 c_3 - c_2^2)\, b^2,$$

$$s_0 = 2\, c_1^{-2/3}(c_1\, c_3 - c_2^2) / \{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\},$$

$$s_1 = \alpha^{-1}\, c_1^{1/3}\, c_2 / \{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\},$$

and $$s_2 = -\alpha^{-2}\, c_1^{-2/3}\, c_2^2\, b^2 / \{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\}.$$

Therefore from the equation (2.11), we get

$$g^{ij} = c_1^{-2/3}\, a^{ij} - [2\,(c_1\, c_3 - c_2^2)/c_1^{2/3}\, \{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\}]\, b^i\, b^j$$
$$+ [c_1^{1/3}\, c_2 / \alpha\{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\}]\, (b^i\, y^j + b^j\, y^i)$$
$$+ [c_2^2\, b^2 / c_1^{2/3}\, \alpha^2\{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\}]\, y^i\, y^j. \qquad (3.4)$$

By using equation (3.1) and (3.4), we have

$$g^{ij}\, b_i\, b_j = b^2\, c_1^{4/3} / \{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\}.$$

Hence, we get

$$b_i(x) = [b^2\, c_1^{4/3} / \{c_1^2 + 2(c_1\, c_3 - c_2^2)\, b^2\}]^{1/2}\, N_i, \quad \text{where } b^2 = a^{ij}\, b_i\, b_j . \qquad (3.5)$$

Hence from (3.4) and (3.5), we can write

$$b^i = a^{ij}\, b_j = [(b^2\, c_1^{-4/3})\{c_1^2 + 2(c_1\, c_3 - c_2^2) b^2\}]^{1/2}\, N^i + (c_2\, b^2/\alpha c_1)\, y^i.$$

Hence, we have the following :

Theorem (3.1). Let F^n be a Finsler space with (α, β) metric (1.2) and $b_i(x) = \partial_i b(x)$ and $F^{n-1}(c)$ be a hypersurface of F^n given by $b(x) = c$ (constant). If the Riemannian metric $a_{ij}(x)\, dx^i\, dx^j$ be positive definite and b_i is a non-zero field, then the induced metric of $F^{n-1}(c)$ is a Riemannian metric given by (3.2) and relations (3.5) and (3.6) hold.

References

1. Hashiguchi, S. Hojo and . Matsumoto, M. : On Landsberg spaces of two dimensions with (α, β) metric, J. Korean Math. Soc., 10 (1973), 17-26.

2. Matsumoto, M. : On C-reducible Finsler spaces, Tensor, N. S., 24 (1972), 29-37.

3. Matsumoto, M. : The induced and intrinsic Finsler connection of a hypersurface and Finslerian projective geometry, J. Math. Kyoto Univ., 25 (1985), 107-144.

4. Park, Hong Sun and Choi, Eun Soo : On a Finsler space with a special (α, β) metric.