

ON THE BI-RECURRENT BOCHNER CURVATURE TENSOR

By

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ABSTRACT : In the present paper, we have studied bi-recurrent and bi-symmetric properties of Bochner Curvature tensor and several theorems have been established.

1. INTRODUCTION

The projective curvature tensor of an n -dimensional Riemannian space M^n is given by

$$W_{ijk}^h = R_{ijk}^h + \frac{1}{(n-1)}(R_{ik} \delta_j^h - R_{jk} \delta_i^h) \quad (1.1)$$

which is invariant under any projective correspondence, where R_{ijk}^h and R_{ik} are the Riemannian curvature and Ricci tensor.

The conformal curvature tensor of M^n is given by

$$C_{ijk}^h = R_{ijk}^h + \frac{1}{(n-1)}(R_{jk}^i \delta_j^h - R_{jk} \delta_j^h + g_{ik} R_j^h - g_{jk} R_i^h) - \frac{R}{(n-1)(n-2)}(g_{ik} \delta_j^h - g_{jk} \delta_i^h) \quad (1.2)$$

where g_{ij} is the Riemannian metric of M^n and

$$R_i^h = g^{ha} R_{ia}, \quad R = g^{ij} R_{ij}$$

Let K^n be an $n(=2m)$ dimensional Kaehlerian space with the structure tensor g_{ij} and F_i^h . It is known that tensor

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)} (R_{ik} \delta_j^h - R_{jk} \delta_i^h + S_{ik} F_j^h - S_{jk} F_i^h + 2S_{ij} F_k^h) \quad (1.3)$$

Called the holomorphically Projective curvature tensor of K^n , is invariant under any holomorphically Projective correspondence (Tachibana and Ishihara [6]). P_{ijk}^h may be considered as the tensor corresponding to W_{ijk}^h .

On the other hand, (S. Bochner [4], Yano and Bochner [5]) has introduced a tensor in K^n given by

$$K_{hijk} = R_{hijk} - \frac{1}{(n+2)} (R_{ik} g_{hk} + R_{hi} g_{jk} + g_{ij} R_{hk} + g_{hi} R_{jk}) \\ + \frac{R}{2(n+1)(n+2)} (g_{ij} g_{hk} + g_{hi} g_{jk})$$

with respect to complex local coordinate.

2. KAEHLERIAN SPACE

An $n(=2m)$ dimensional Kaehlerian space K^n is a Riemannian space which admits a tensor field F_i^j satisfying the conditions

$$F_a^j F_j^a = -\delta_j^j \quad (2.1)$$

$$F_{ij} = -F_{ji} \quad (F_i^a = F_i^a g_{aj}) \quad (2.2)$$

$$\text{and } F_{i,k}^j = 0 \quad (2.3)$$

where the comma(,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ik} of the Riemannian space.

The Riemannian curvature tensor R_{ijk}^h is given by

$$R_{ijk}^h = \partial_i \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ik \end{matrix} \right\} + \left\{ \begin{matrix} h \\ ia \end{matrix} \right\} \left\{ \begin{matrix} a \\ jk \end{matrix} \right\} - \left\{ \begin{matrix} h \\ ja \end{matrix} \right\} \left\{ \begin{matrix} a \\ ik \end{matrix} \right\} \quad (2.4)$$

where $\partial_i = \frac{\partial}{\partial x^i}$ and $\{x^i\}$ denotes the real local coordinates.

The Ricci tensor and the scalar curvature tensor are respectively given by

$$R_{jk} = R_{ajk}^a \quad \text{and} \quad R = g^{ij} R_{ij}$$

It is well known that these tensors satisfy the following identities

$$R_{ajk}^h F_i^a = -R_{iak}^h F_j^a \quad (2.5)$$

$$R_{ijk}^h F_k^a = R_{ijka}^h F_a^h \quad (2.6)$$

$$F_i^a R_{aj} = -R_{ia} F_j^a \quad (2.7)$$

$$F_i^a R_i^h = R_i^a F_a^h \quad (2.8)$$

$$R_{ijk,a}^a = R_{jk,i} - R_{ik,j} \quad (2.9)$$

and $R_j = 2R_{i,a}^a \quad (2.10)$

If we define a tensor S_{jk} by

$$S_{jk} = F_j^a R_{ak} \quad (2.11)$$

then we have

$$S_{jk} = -S_{kj} \quad (2.12)$$

$$F_i^a S_{ak} = -S_{ia} F_k^a \quad (2.13)$$

$$S_{jk} = \tau \left(\frac{1}{2} \right) F^{ab} R_{abjk} \quad (2.14)$$

$$2S_{i,a}^a = F_i^a R_{,a} \quad (2.15)$$

Now we shall consider a tensor K_{ijk}^h defined by

$$\begin{aligned}
K_{ijk}^h &= R_{ijk}^h + \frac{1}{(n+4)} (R_{ik} \delta_j^h - R_{jk} \delta_i^h + g_{ik} R_j^h - g_{jk} R_i^h + S_{ik} F_j^h \\
&\quad - S_{jk} F_i^h + F_{ik} S_j^h - F_{jk} S_i^h + 2S_{ij} F_k^h + 2F_{ij} S_k^h) \\
&\quad - \frac{R}{(n+2)(n+4)} (g_{ik} \delta_j^h - g_{jk} \delta_i^h + F_{ik} F_j^h - F_{jk} F_i^h + 2F_{ij} F_k^h) \quad (2.16)
\end{aligned}$$

which is constructed formally from C_{ijk}^h by taking account of the formal resemblance between W_{ijk}^h and P_{ijk}^h . Then we can prove that the tensor $K_{ijk a}^h = g_{ha} K_{ijk}^h$ has components of tensor given by S. Bochner with respect to complex local coordinates.

Hence we call tensor the Bochner curvature tensor.

Now if we put

$$L_{ij} = R_{ij} - \frac{R}{2(n+2)} g_{ij} \quad \text{and} \quad (2.17a)$$

$$M_{ij} = F_i^a L_{aj} = S_{ij} - \frac{R}{2(n+2)} F_{ij} \quad (2.17b)$$

then (2.16) reduces to the form

$$\begin{aligned}
K_{ijk}^h &= R_{ijk}^h + \frac{1}{(n+4)} (L_{ik} \delta_j^h - L_{jk} \delta_i^h + g_{ik} L_j^h - g_{jk} L_i^h + M_{ik} F_j^h - M_{jk} F_i^h \\
&\quad + F_{ik} M_j^h - F_{jk} M_i^h + 2M_{ij} F_k^h + 2F_{ij} M_k^h) \quad (2.18)
\end{aligned}$$

The following identities are obtained by the straight forward computations.

$$K_{ijk}^h = -K_{jki}^h, \quad K_{ijk a}^h = -K_{ijak}^h$$

$$K_{ijk}^h + K_{jki}^h + K_{kij}^h = 0$$

$$K_{ajk}^a = 0, \quad K_{ija}^a = 0$$

$$K_{ijk}^a F_a^h = K_{ija}^h F_k^a, \quad K_{ajk}^h F_i^a = -K_{iak}^h F_j^a,$$

$$K_{ija}^b F_b^a = 0, \quad K_{ajk}^b F_b^a = 0 \quad (*1)$$

Now we shall use the following definitions.

DEFINITION 2.1 : A Kaehler space satisfying the relation

$$R_{ijk,ab}^h - \lambda_{ab} R_{ijk}^h = 0 \quad (2.19)$$

for some non-zero tensor λ_{ab} , will be called a Kaehlerian bi-recurrent space and the space is called Ricci bi-recurrent (or semi bi-recurrent) if it satisfies

$$R_{ij,ab} - \lambda_{ab} R_{ij} = 0 \quad (2.20)$$

Multiplying the above equation by g^{ij} , we have

$$R_{,ab} - \lambda_{ab} R = 0 \quad (2.21)$$

DEFINITION 2.2 : A Kaehler space satisfying the relation

$$P_{ijk,qb}^h - \lambda_{ab} P_{ijk}^h = 0 \quad (2.22)$$

for some non-zero tensor λ_{ab} , will be called a Kaehlerian space with bi-recurrent H-projective curvature tensor or briefly K-P space.

DEFINITION 2.3 : A Kaehler space satisfying the relation

$$B_{ijk,ab}^h - \lambda_{ab} B_{ijk}^h = 0 \quad (2.23)$$

for some non-zero tensor λ_{ab} , will be called a Kaehlerian space with bi-recurrent Bochner curvature tensor or briefly K-B space.

THEOREM 1 : The necessary and sufficient condition that a Kaehlerian space is Kaehlerian Ricci bi-recurrent is that

$$K_{ijk,ab}^h - \lambda_{ab} K_{ijk}^h = R_{ijk,ab}^h - \lambda_{ab} R_{ijk}^h$$

(*1) The holomorphically projective curvature tensor satisfies these all identities except the second one see, S. Tachibana and S. Ishihara [6].

PROOF : Let the space be Kaehlerian Ricci bi-recurrent, then the relation (2.20), is satisfied.

Differentiating (2.18), covariantly with respect to x^a and again differentiating the result thus obtained with respect to x^b , we get

$$K_{ijk,ab}^h = R_{ijk,ab}^h + \frac{1}{(n+4)} (L_{ik,ab} \delta_j^h - L_{jk,ab} \delta_i^h + g_{ik} L_{j,ab}^h - g_{jk} L_{i,ab}^h + M_{ik,ab} F_j^h - M_{jk,ab} F_i^h + F_{ik,ab} M_j^h - F_{jk,ab} M_i^h + 2M_{i,j,ab} F_k^h + 2F_{i,j,ab} M_k^h) \quad (2.24)$$

Transvecting (2.18) with λ_{ab} , and subtracting the result thus obtained from (2.24), we get

$$K_{ijk,ab}^h - \lambda_{ab} K_{ijk}^h = R_{ijk,ab}^h - \lambda_{ab} R_{ijk}^h + \frac{1}{(n+4)} [(L_{ik,ab} - \lambda_{ab} L_{ik}) \delta_j^h - (L_{jk,ab} - \lambda_{ab} L_{jk}) \delta_i^h + g_{ik} (L_{j,ab}^h - \lambda_{ab} L_j^h) - g_{jk} (L_{i,ab}^h - \lambda_{ab} L_i^h) + (M_{ik,ab} - \lambda_{ab} M_{ik}) F_j^h - (M_{jk,ab} - \lambda_{ab} M_{jk}) F_i^h + (F_{ik,ab} - \lambda_{ab} F_{ik}) M_j^h - (F_{jk,ab} - \lambda_{ab} F_{jk}) M_i^h + 2F_k^h (M_{i,j,ab} - \lambda_{ab} M_{ij}) + 2M_k^h (F_{i,j,ab} - \lambda_{ab} F_{ij})] \quad (2.25)$$

Since the space is Ricci bi-recurrent then the equation (2.25), in view of (2.17a) and (2.17b), reduces to

$$K_{ijk,ab}^h - \lambda_{ab} K_{ijk}^h = R_{ijk,ab}^h - \lambda_{ab} R_{ijk}^h \quad (2.26)$$

Conversely, if in a Kaehlerian space equation (2.26), is satisfied then we have from (2.25),

$$[(L_{ik,ab} - \lambda_{ab} L_{ik}) \delta_j^h - (L_{jk,ab} - \lambda_{ab} L_{jk}) \delta_i^h + g_{ik} (L_{j,ab}^h - \lambda_{ab} L_j^h) - g_{jk} (L_{i,ab}^h - \lambda_{ab} L_i^h) + (M_{ik,ab} - \lambda_{ab} M_{ik}) F_j^h - (M_{jk,ab} - \lambda_{ab} M_{jk}) F_i^h + (F_{ik,ab} - \lambda_{ab} F_{ik}) M_j^h - (F_{jk,ab} - \lambda_{ab} F_{jk}) M_i^h + 2F_k^h (M_{i,j,ab} - \lambda_{ab} M_{ij}) + 2M_k^h (F_{i,j,ab} - \lambda_{ab} F_{ij})] = 0 \quad (2.27)$$

which yields with the help of (2.17a) and (2.17b),

$$R_{ijk,ab}^h - \lambda_{ab} R_{ijk}^h = 0$$

i.e. the space is Kaehlerian Ricci bi-recurrent.

THEOREM 2 : Every Kaehlerian bi-recurrent space is a Kaehlerian space with bi-recurrent Bochner curvature tensor.

PROOF : If the space is Kaehlerian bi-recurrent, then equations (2.19) and (2.20) are satisfied and equation (2.25), in view of (2.19), (2.20), (2.17a) and (2.17b) reduces to

$$K_{ijk,ab}^h - \lambda_{ab} K_{ijk}^h = 0$$

which shows that the space will also be Kaehlerian space with bi-recurrent Bochner curvature tensor.

Next we consider a tensor K_{ijk} , given by

$$K_{ijk} = R_{ijk} - R_{ikj} + \frac{R_i}{2(n+2)} (g_{ik} \delta_j^h - g_{jk} \delta_i^h + F_{ik} F_j^l - F_{jk} F_i^l + 2F_{ij} F_k^l)$$

then we can get the following identity

$$K_{ijk,a}^a = \frac{n}{n+4} K_{ijk}$$

Now we consider a tensor

$$U_{ijk}^h = R_{ijk}^h + \frac{R}{n(n+2)} (g_{ik} \delta_j^h - g_{jk} \delta_i^h + F_{ik} F_j^h - F_{jk} F_i^h + 2F_{ij} F_k^h)$$

then we can obtain the following.

THEOREM 3 : The Bochner curvature tensor coincides with U_{ijk}^h of a Kaehlerian space K^n is an Einstein space.

Remark : The term of a Riemannian space defined by

$$Z_{ijk}^h = R_{ijk}^h + \frac{R}{n(n-1)} (g_{ik} \delta_j^h - g_{jk} \delta_i^h)$$

Is called the concircular curvature tensor and is invariant under any concircular correspondence (by Yano [2]) U_{ijk}^h correspondence to Z_{ijk}^h . A Kaehlerian space is called a space of constant holomorphically sectional curvature if its U_{ijk}^h vanishes identically.

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