

T3-Like Finsler Spaces

S. C. Rastogi

Set Vishambhar Nath Institute of Engineering, Research and Technology,
Safedabad, Barabanki-225003, U.P., India

e-mail : sureshrastogi@rediffmail.com

(Received : April 21, 2008)

Abstract

In 1972, T-tensor in a Finsler space of n-dimensions was introduced and studied simultaneously by H. Kawaguchi [3] and Matsumoto [5]. Several papers related with T-tensor, since then, have been published by various authors namely Hashiguchi [1], Matsumoto [6], Matsumoto and Shimada [7, 8], Rastogi [10, 11] and others. The purpose of the present paper is to study some properties of T-tensor in a Finsler space of three dimensions. Furthermore, we have defined and studied Finsler spaces F_n , whose T-tensor is of special form and called them T3-like Finsler spaces.

1. Introduction

Let (l^i, m^i, n^i) be the Moor's frame of a Finsler space of three dimensions F^3 , where l^i is normalized supporting element : $y^i = L l^i$, m^i is normalized torsion vector given by $C^i = C m^i$ and n^i is a unit vector orthogonal to both m^i and l^i . The metric and angular metric tensors in F^3 are given by [6] :

$$g_{ij} = l_i l_j + m_i m_j + n_i n_j, \quad h_{ij} = m_i m_j + n_i n_j \quad (1.1)$$

while the Cartan's C-tensor is given by Matsumoto [6] as follows :

$$C_{ijk} = C_{(1)} m_i m_j m_k - C_{(2)} (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) \\ + C_{(3)} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + C_{(2)} n_i n_j n_k. \quad (1.2)$$

The T-tensor in a Finsler space of n-dimensions is given by

$$T_{ijkh} = L C_{ijk} l_h + l_i C_{jkh} + l_j C_{ikh} + l_k C_{ijh} + l_h C_{ijk}, \quad (1.3)$$

where l_h denotes v -covariant derivative Matsumoto [6].

2. Cartan's C-Tensor in F^3

Let $C_{ijk} m^k = 'C_{ij}$ and $C_{ijk} n^k = *C_{ij}$, then from equation (1.2) we can obtain

$$'C_{ij} = C_{(1)} m_i m_j - C_{(2)} (m_i n_j + m_j n_i) + C_{(3)} n_i n_j \quad (2.1)$$

and
$$*C_{ij} = C_{(2)} (n_i n_j - m_i m_j) + C_{(3)} (m_i n_j + m_j n_i), \quad (2.2)$$

i.e., these tensors are symmetric in lower indices.

Further from equations (2.1) and (2.2), we can obtain

$$'C_{ij} m^j = C_{(1)} m_i - C_{(2)} n_i, \quad *C_{ij} m^j = -C_{(2)} m_i + C_{(3)} n_i, \quad (2.3)$$

$$'C_{ij} m^j m^i = C_{(1)}, \quad 'C_{ij} m^j n^i = -C_{(2)}, \quad 'C_{ij} n^j n^i = C_{(3)} \quad (2.4)$$

and
$$*C_{ij} m^j n^i = -C_{(2)}, \quad *C_{ij} m^j m^i = C_{(3)}, \quad *C_{ij} n^j n^i = C_{(2)}. \quad (2.5)$$

In a three dimensional Finsler space F^3 , v -covariant derivatives of the vectors l^i , m^i and n^i are given in Matsumoto [6] as follows :

$$Ll^i_j = h^i_j, \quad Lm^i_j = -l^i m_j + n^i v_j, \quad Ln^i_j = -l^i n_j - m^i v_j \quad (2.6)$$

where $v_j = v_{23\gamma} e_{\gamma j}$, therefore from equations (2.1) and (2.2) we can respectively obtain

$$\begin{aligned} 'C_{ij} l_h = & C_{(1)} l_h m_i m_j - C_{(2)} l_h (m_i n_j + m_j n_i) + C_{(3)} l_h n_i n_j \\ & + L^{-1} [C_{(1)} \{m_i (-l_j m_h + n_j v_h) + m_j (-l_i m_h + n_i v_h)\} \\ & + C_{(2)} \{m_i (l_j n_h + m_j v_h) + m_j (l_i n_h + m_i v_h) + n_i (l_j m_h - n_j v_h) \\ & + n_j (l_i m_h - n_i v_h)\} - C_{(3)} \{n_i (l_j n_h + m_j v_h) + n_j (l_i n_h + m_i v_h)\}] \quad (2.7) \end{aligned}$$

and

$$\begin{aligned} *C_{ij} l_h = & C_{(2)} l_h n_i n_j - m_i m_j + C_{(3)} l_h (m_i n_j + m_j n_i) \\ & + L^{-1} [C_{(2)} \{m_i (l_j m_h - n_j v_h) + m_j (l_i m_h - n_i v_h) \\ & - n_i (l_j n_h + m_j v_h) - n_j (l_i n_h + m_i v_h)\} - C_{(3)} \{m_i (l_j n_h + m_j v_h) \end{aligned}$$

$$+ m_j (l_i n_h + m_i v_h) + n_i (l_j m_h - n_j v_h) + n_j (l_i n_h - n_i v_h) \}. \quad (2.8)$$

Equations (2.7) and (2.8) respectively give

$$\begin{aligned} 'C_{ij|h} l^j + L^{-1} 'C_{ih} &= 0, \\ 'C_{ij|h} l^h &= C_{(1)0} m_i m_j - C_{(2)0} (m_i n_j + m_j n_i) + C_{(3)0} n_i n_j, \\ 'C_{ij|h} m^j &= L^{-1} l_i (C_{(2)0} n_h - C_{(1)0} m_h) + m_i (C_{(1)h} + 2L^{-1} C_{(2)0} v_h) \\ &\quad + n_i \{L^{-1} (C_{(1)0} - C_{(3)0}) v_h - C_{(2)h}\}, \\ 'C_{ij|h} n^j &= -L^{-1} l_i (C_{(2)0} m_h + C_{(3)0} n_h) + m_i \{L^{-1} (C_{(1)0} - C_{(3)0}) v_h - C_{(2)h}\} \\ &\quad + n_i (C_{(3)h} - 2L^{-1} C_{(2)0} v_h) \\ *C_{ij|h} l^j + L^{-1} *C_{ih} &= 0, \\ *C_{ij|h} l^h &= C_{(2)0} (n_i n_j - m_i m_j) + C_{(3)0} (m_i n_j + m_j n_i), \\ *C_{ij|h} m^j &= L^{-1} l_i (C_{(2)0} m_h - C_{(3)0} n_h) - m_i (C_{(2)h} + 2L^{-1} C_{(3)0} v_h) \\ &\quad + n_i (C_{(3)h} - 2L^{-1} C_{(2)0} v_h), \\ *C_{ij|h} n^j &= -L^{-1} l_i (C_{(2)0} n_h + C_{(3)0} m_h) + m_i (C_{(3)h} - 2L^{-1} C_{(2)0} v_h) \\ &\quad + n_i (C_{(2)h} + 2L^{-1} C_{(3)0} v_h) \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} 'C_{ij|h} m^j m^h m^i &= 2L^{-1} C_{(2)0} v_{2)32} + C_{(1)h} m^h, \\ 'C_{ij|h} m^j m^h n^i &= L^{-1} (C_{(1)0} - C_{(3)0}) v_{2)32} - C_{(2)h} m^h, \\ 'C_{ij|h} m^j n^h m^i &= 2L^{-1} C_{(2)0} v_{2)33} + C_{(1)h} n^h, \\ 'C_{ij|h} m^j n^h n^i &= L^{-1} (C_{(1)0} - C_{(3)0}) v_{2)323} - C_{(2)h} n^h, \\ *C_{ij|h} m^j m^h m^i &= -2L^{-1} C_{(3)0} v_{2)32} - C_{(2)h} m^h, \\ *C_{ij|h} m^j m^h n^i &= -2L^{-1} C_{(2)0} v_{2)32} + C_{(3)h} m^h, \\ *C_{ij|h} m^j n^h m^i &= -2L^{-1} C_{(3)0} v_{2)33} - C_{(2)h} n^h, \end{aligned}$$

$$*C_{ij|h}^l m^j n^h n^i = -2L^{-1} C_{(2)} v_{2)33} + C_{(3)}^l|_h n^h. \quad (2.10)$$

Since $C_{(1)} + C_{(3)} = C$, therefore from equation (2.10), we can obtain

$$\{('C_{ij|h}^l m^i + *C_{ij|h}^l n^i) m^j - C|_h\} m^h = 0, \quad (2.11)$$

$$\{('C_{ij|h}^l n^i + *C_{ij|h}^l m^i) m^j m^h - L^{-1} C_{(1)} v_{2)32}\} \quad (2.12)$$

and

$$2 *C_{ij|h}^l m^j (C_{(3)} n^i - C_{(2)} m^i) = (C_{(2)}^2 + C_{(3)}^2)|_h. \quad (2.13)$$

Hence, we have :

Theorem (2.1). In a three dimensional Finsler space F^3 , $'C_{ij|h}^l$ and $*C_{ij|h}^l$ satisfy equations (2.11), (2.12) and (2.13).

If we take h -covariant derivative of equations (2.1) and (2.2) and use $l^i_j = 0$, $m^i_j = n^i h_j$, $n^i_j = -m^i h_j$, we get

$$\begin{aligned} 'C_{ijkl} = & (C_{(1)lk} + 2C_{(2)} h_k) m_i m_j + (C_{(3)lk} - 2C_{(2)} h_k) n_i n_j \\ & + \{(C_{(1)} - C_{(3)}) h_k - C_{(2)lk}\} (m_i n_j + m_j n_i) \end{aligned} \quad (2.14)$$

and

$$*C_{ijkl} = (C_{(3)lk} - C_{(2)} h_k) (n_i m_j + m_i n_j) + (C_{(2)lk} + 2C_{(3)} h_k) (n_i n_j - m_i m_j), \quad (2.15)$$

which leads to

$$(*C_{ijkl} m^j - 'C_{ijkl} n^j) n^i - C_{(2)} h_k = 0, \quad (2.16)$$

$$'C_{ijkl} (m^j m^i + n^j n^i) = C_{lk}, \quad (2.17)$$

and

$$'C_{ijkl} m^j n^i - *C_{ijkl} m^j m^i = 'C_{ijkl} m^j n^i + *C_{ijkl} n^j n^i = C h_k. \quad (2.18)$$

Hence, we have

Theorem (2.2). In a three dimensional Finsler space F^3 , h -covariant derivative of $'C_{ij}$ and $*C_{ij}$ satisfy equations (2.16), (2.17) and (2.18).

In case of a P^* -Finsler space F^3 , Izumi [2], $P_{ijk} = \lambda A_{ijk}$, therefore we can obtain

$$'C_{ij|0} = \lambda 'C_{ij} + h_0 *C_{ij}, \quad *C_{ij|0} = \lambda *C_{ij} - h_0 'C_{ij}. \quad (2.19)$$

Multiplying equation (2.19) by m^j and using (2.1) and (2.2), we can obtain on simplification

$$\begin{aligned} C_{(1)0} &= \lambda C_{(1)} - 3h_0 C_{(2)}, & C_{(2)0} &= \lambda C_{(2)} + (C_{(1)} - 2C_{(3)}) h_0, \\ C_{(3)0} &= \lambda C_{(3)} + 3h_0 C_{(2)}. \end{aligned} \quad (2.20)$$

Hence, we have

Theorem (2.3). In a three dimensional P*-Finsler space F^3 , $C_{(1)0}$, $C_{(2)0}$ and $C_{(3)0}$ are respectively given by (2.20).

Furthermore, from equation (2.19), we can obtain with the help of equations (2.4) and (2.5)

$$\begin{aligned} 'C_{ij0} m^j m^i &= C_{(1)0} + 2C_{(2)} h_0, & 'C_{ij0} m^j n^i &= -C_{(2)0} + h_0(C_{(1)} - C_{(3)}), \\ 'C_{ij0} n^j n^i &= C_{(3)0} - 2C_{(2)} h_0. \end{aligned} \quad (2.21)$$

Hence, we have

Theorem (2.4). In a three dimensional P*-Finsler space F^3 , $'C_{ij0}$ satisfies equation (2.21).

Further from equation (2.19), we can obtain on simplification

$$'C_{ij} = (h_0^2 + \lambda^2)^{-1} (\lambda 'C_{ij0} - h_0 *C_{ij0}) \quad (2.22)$$

and

$$*C_{ij} = (h_0^2 + \lambda^2)^{-1} (\lambda *C_{ij0} + h_0 'C_{ij0}) \quad (2.23)$$

which lead to

Theorem (2.5). In a P*-Finsler space F^3 , $'C_{ij}$ and $*C_{ij}$ respectively satisfy equations (2.22) and (2.23).

3. v-Covariant Derivative of C-Tensor in F^3

Taking v-covariant derivative of (1.2) and using equation (2.5) we can obtain on simplification

$$C_{ijk|h} = C_{(1)h} m_i m_j m_k - C_{(2)h} (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k)$$

$$\begin{aligned}
& + C_{(3)h} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) \\
& - L^{-1} \sum_{(i,j,k)} [C_{(1)} (l_i m_h - n_i v_h) m_j m_k - C_{(2)} \{l_i (m_h m_j n_k + m_j m_k n_h \\
& \quad + m_h m_k n_j - n_h n_j n_k) + v_h (m_i m_j m_k - m_j m_k n_i - 2m_i n_k n_j)\} \\
& \quad + C_{(3)} \{l_i (m_h n_j n_k + m_j n_k n_h + m_k n_h n_j) + 2m_i m_j v_h n_k\}], \quad (3.1)
\end{aligned}$$

which by virtue of symmetry of C_{ijk}^l in k and h easily leads to

$$\begin{aligned}
& \zeta_{(k,h)} \{ [C_{(1)h} m_i m_j m_k - C_{(2)h} (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k) \\
& \quad + C_{(3)h} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)] + L^{-1} [C_{(1)} \{(m_i n_j + m_j n_i) m_k v_h \\
& \quad + m_i m_j (l_h m_k + n_k v_h)\} - C_{(2)} \{(m_i n_j + m_j n_i) (m_k l_h + 3n_k v_h) \\
& \quad + (m_i m_j - n_i n_j) (l_h n_k + 3v_k m_h)\} + C_{(3)} \{(m_i n_j + m_j n_i) (n_k l_h + 2v_k m_h) \\
& \quad + (m_i m_j - n_i n_j) 2v_k n_h + n_i n_j (l_h m_k + v_h n_k)\}] = 0. \quad (3.2)
\end{aligned}$$

Multiplying equation (3.2) by $m^i m^j m^k$, we get on simplification

$$\alpha_h - \alpha_k m^k m_h + \beta_k m^k n_h = 0, \quad (3.3)$$

where α_h and β_h are given by

$$\alpha_h = LC_{(1)h} + C_{(1)} l_h + 3C_{(2)} v_h \quad (3.4)$$

$$\text{and } \beta_h = LC_{(2)h} + C_{(2)} l_h - (C_{(1)} - 2C_{(3)}) v_h. \quad (3.5)$$

With the help of equations (3.3), (3.4) and (3.5), we can obtain

$$\alpha_0 = \alpha_h l^h = LC_{(1)0} + C_{(1)}, \quad \beta_0 = \beta_h l^h = LC_{(2)0} + C_{(2)} \quad (3.6)$$

$$\alpha_h m^h = LC_{(1)h} m^h + 3C_{(2)} v_{2)32}, \quad (3.7)$$

$$\alpha_h n^h = LC_{(1)h} n^h + 3C_{(2)} v_{2)33}, \quad (3.8)$$

$$\beta_h m^h = LC_{(2)h} m^h - (C_{(1)} - 2C_{(3)}) v_{2)32}, \quad (3.9)$$

$$\beta_h n^h = LC_{(2)h} n^h - (C_{(1)} - 2C_{(3)}) v_{2)33}, \quad (3.10)$$

and

$$LC_{(1)h}^l n^h + 3C_{(2)} v_{2)33} + LC_{(2)h}^l m^h - (C_{(1)} - 2C_{(3)}) v_{2)32} = 0. \quad (3.11)$$

Hence, we have

Theorem (3.1). In a three dimensional Finsler space F^n , the coefficients $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ satisfy equations (3.6) to (3.11).

Multiplying equation (3.2) by $n^i n^j n^k$, we get on simplification

$$\begin{aligned} LC_{(2)h}^l + C_{(2)} l_h + 3C_{(3)} v_h - (LC_{(3)k}^l n_k - 3C_{(2)} v_{2)33}) m_h \\ - (LC_{(2)k}^l n^k + 3C_{(3)} v_{2)33}) n_h = 0. \end{aligned} \quad (3.12)$$

Multiplying equation (3.12) by m^h , we get

$$3(C_{(2)} v_{2)33} + C_{(3)} v_{2)32}) = L(C_{(3)h}^l n^h - C_{(2)h}^l m^h). \quad (3.13)$$

Hence, we have

Theorem (3.2). In a three dimensional Finsler space F^n , the coefficients $C_{(2)h}^l$ and $C_{(3)h}^l$ satisfy equation (3.13).

Multiplying equation (3.2) by $n^i n^k m^j m^h$, we get on simplification

$$3C_{(2)} v_{2)32} + (C_{(1)} - 2C_{(3)}) v_{2)33} = L(C_{(3)h}^l m^h + C_{(2)h}^l n^h). \quad (3.14)$$

Hence, we have

Theorem (3.3). In a three dimensional Finsler space F^3 , the coefficients $C_{(2)h}^l$ and $C_{(3)h}^l$ satisfy equation (3.14).

From equations (3.13) and (3.14), on simplification we can obtain

$$\begin{aligned} L[C_{(2)} C_{(3)} \{\log(C_{(3)} C_{(2)}^{-1})\} l_h n^h - (1/2)(C_{(2)}^2 + C_{(3)}^2) l_h m^h] \\ = v_{2)33} (3C_{(2)}^2 + 2C_{(3)}^2 - C_{(1)} C_{(3)}). \end{aligned} \quad (3.15)$$

Hence, we have

Theorem (3.4). In a three dimensional Finsler space F^n , the coefficients $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ satisfy equation (3.15).

Multiplying equation (3.1) by $g^{ij}(x, y)$ and using $C_{(1)} + C_{(3)} = C$, we can obtain on

simplification

$$C_k^j|_h = C|_h m_k - L^{-1} C \{L_k m_h - (1/4) n_k v_h\}. \quad (3.16)$$

Hence, we have

Theorem (3.5). In a three dimensional Finsler space F^3 , scalar C satisfies (3.16).

4. P-Tensor in F^3

In a three dimensional Finsler space F^3 , P-tensor is expressed as

$$\begin{aligned} P_{ijk} = & L[(C_{(1)|0} + 3C_{(2)} h_0) m_i m_j m_k - \{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} \\ & (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) + (C_{(3)|0} - 3C_{(2)} h_0) \\ & (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + (C_{(2)|0} + 3C_{(3)} h_0) n_i n_j n_k]. \end{aligned} \quad (4.1)$$

Let $P_{ijk} m^k = {}^r P_{ij}$ and $P_{ijk} n^k = {}^* P_{ij}$, then from equation (4.1), we can obtain

$$\begin{aligned} {}^r P_{ij} = & L[(C_{(1)|0} + 3C_{(2)} h_0) m_i m_j - \{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} \\ & (m_j n_i + m_i n_j) + (C_{(3)|0} - 3C_{(2)} h_0) n_i n_j] \end{aligned} \quad (4.2)$$

and

$$\begin{aligned} {}^* P_{ij} = & L[-\{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} m_i m_j + (C_{(3)|0} - 2C_{(2)} h_0) \\ & (m_i n_j + m_j n_i) + (C_{(2)|0} + 3C_{(3)} h_0) n_i n_j] \end{aligned} \quad (4.3)$$

which are symmetric tensors in lower indices. Further from equations (4.2) and (4.3), we can obtain

$${}^r P_{ij} m^j = L[(C_{(1)|0} + 3C_{(2)} h_0) m_i - \{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} n_i] \quad (4.4)$$

$${}^r P_{ij} n^j = L[-\{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} m_i + (C_{(3)|0} - 3C_{(2)} h_0) n_i] \quad (4.5)$$

$${}^r P_{ij} m^j m^i = L(C_{(1)|0} + 3C_{(2)} h_0), \quad {}^r P_{ij} m^j n^i = -L\{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} \quad (4.6)$$

$${}^r P_{ij} n^j n^i = L(C_{(3)|0} - 3C_{(2)} h_0), \quad (4.7)$$

$${}^* P_{ij} m^j = L[-\{C_{(2)|0} - (C_{(1)} - 2C_{(3)}) h_0\} m_i + (C_{(3)|0} - 3C_{(2)} h_0) n_i], \quad (4.8)$$

$${}^* P_{ij} n^j = L[(C_{(3)|0} - 3C_{(2)} h_0) m_i + (C_{(2)|0} + 3C_{(3)} h_0) n_i], \quad (4.9)$$

$$*P_{ij} m^j m^i = -L\{C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0\}, \quad (4.10)$$

$$*P_{ij} m^j n^i = L(C_{(3)0} - 3C_{(2)} h_0), \quad (4.11)$$

$$*P_{ij} n^j n^i = L(C_{(2)0} + 3C_{(3)} h_0). \quad (4.12)$$

From $C_{ijk} m^k = {}'C_{ij}$ and $C_{ijk} n^k = {}^*C_{ij}$, we can easily obtain

$${}'P_{ij} = L({}'C_{ij0} - h_0 {}^*C_{ij}), \quad {}^*P_{ij} = L({}^*C_{ij0} + h_0 {}'C_{ij}). \quad (4.13)$$

Hence, we have

Theorem (4.1). In a three dimensional Finsler space, tensors ${}'P_{ij}$ and ${}^*P_{ij}$ are related with ${}'C_{ij}$ and ${}^*C_{ij}$ by equation (4.13).

In case of a P*-Finsler space F^3 , with the help of equation (2.19) and (4.13), we can establish

Theorem (4.2). In a three dimensional P*-Finsler space F^3 , ${}'P_{ij} = L\lambda {}'C_{ij}$ and ${}^*P_{ij} = L\lambda {}^*C_{ij}$.

5. T-Tensor in F^3

Substituting the value of C_{ijk} and $C_{ijk}^l{}_h$ from equations (1.2) and (3.1) in (1.3), we can obtain

$$\begin{aligned} T_{ijk} = & L[C_{(1)h}^l m_i m_j m_k - C_{(2)h}^l (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k) \\ & + C_{(3)h}^l (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)] \\ & + C_{(1)} [m_i m_j m_k l_h + v_h (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j)] \\ & - C_{(2)} [v_h (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j - 3 m_i m_j m_k) \\ & + l_h \{n_i n_j n_k - (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j)\}] \\ & + C_{(3)} \{l_h (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) \\ & - 2 v_h (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j)\}. \end{aligned} \quad (5.1)$$

Equation (5.1) can also be expressed as

$$T_{ijkh} = m_i m_j m_k \alpha_h - \sum_{(i,j,k)} \{m_i m_j n_k \beta_h - m_i n_j n_k \gamma_h\} + n_i n_j n_k \delta_n \quad (5.2)$$

where

$$\gamma_h = LC_{(3)}|_h + C_{(3)} l_h + C_{(2)} v_h, \quad \delta_h = LC_{(2)}|_h + C_{(2)} l_h \quad (5.3)$$

such that

$$\begin{aligned} \gamma_0 = \gamma_h l^h &= LC_{(3)}|_0 + C_{(3)}, & \delta_0 = \delta_h l^h &= \beta_0 = LC_{(2)}|_0 + C_{(2)} \\ \gamma_h m^h &= LC_{(3)}|_h m^h + C_{(2)} v_{232}, & \delta_h m^h &= LC_{(2)}|_h m^h \\ \gamma_h n^h &= LC_{(3)}|_h n^h + C_{(2)} v_{233}, & \delta_h n^h &= LC_{(2)}|_h n^h \end{aligned} \quad (5.4)$$

Multiplying equation (5.2) by $g_{jk} l^h$, we can obtain $LC|_0 = -C$, $LC_{(1)}|_0 = -C_{(1)}$ and $LC_{(2)}|_0 = -C_{(2)}$. Hence, we have

Theorem (5.1). In a three dimensional Finsler space F^3 , if T-tensor is expressed by (5.2), coefficients α_0 , β_0 , γ_0 , δ_0 , $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ satisfy $\alpha_0 + \gamma_0 = 0$, $\beta_0 = \delta_0$, $LC|_0 = -C$, $LC_{(1)}|_0 = -C_{(1)}$ and $LC_{(2)}|_0 = -C_{(2)}$.

From equation (5.2), we can obtain by virtue of $T_{ijkh} g^{kh} = T_{ij}$

$$\begin{aligned} T_{ij} &= \{LC_{(1)}|_h m^h + 3C_{(2)} v_{232} - LC_{(2)}|_h n^h + (C_{(1)} - 2C_{(3)}) v_{233}\} m_i m_j \\ &\quad + \{LC_{(3)}|_h n^h + C_{(2)} v_{233} - LC_{(2)}|_h m^h + (C_{(1)} - 2C_{(3)}) v_{232}\} \\ &\quad (m_i n_j + m_j n_i) + \{LC_{(3)}|_h m^h + C_{(2)} v_{232} + LC_{(2)}|_h n^h\} n_i n_j \end{aligned} \quad (5.5)$$

which satisfies $T_{ij} l^i = 0$,

$$\begin{aligned} T_{ij} m^i &= \{LC_{(1)}|_h m^h + 3C_{(2)} v_{232} - LC_{(2)}|_h n^h + (C_{(1)} - 2C_{(3)}) v_{233}\} m_j \\ &\quad + \{LC_{(3)}|_h n^h + C_{(2)} v_{233} - LC_{(2)}|_h m^h + (C_{(1)} - 2C_{(3)}) v_{232}\} n_j \end{aligned} \quad (5.6)$$

and

$$\begin{aligned} T_{ij} n^i &= \{LC_{(3)}|_h m^h + C_{(2)} v_{232} + LC_{(2)}|_h n^h\} n_j \\ &\quad + \{LC_{(3)}|_h n^h + C_{(2)} v_{233} - LC_{(2)}|_h m^h + (C_{(1)} - 2C_{(3)}) v_{232}\} m_j \end{aligned} \quad (5.7)$$

From equation (5.5), we can further obtain by virtue of $T_{ij} g^{ij} = T$

$$T = LC|_h m^h + 4C_{(2)} v_{232} + (C_{(1)} - 2C_{(3)}) v_{233} \quad (5.8)$$

6. T3-Like Finsler Spaces

From equation (5.2), we can write tensor T_{ijkh} of a 3-dimensional Finsler space F^3 in the following form

$$T_{ijkh} = \Sigma_{(i,j,k)} \{a_{hk} h_{ij} + b_{hk} m_i m_j\}, \quad (6.1)$$

where a_{hk} and b_{hk} are the second order tensors defined by

$$a_{hk} = \gamma_h m_k + (1/3) \delta_h n_k, \quad b_{hk} = ((1/3) \alpha_h - \gamma_h) m_k - (\beta_h + (1/3) \delta_h) n_k. \quad (6.2)$$

From equation (6.2), we can obtain

$$a_{0k} = \gamma_0 m_k + (1/3) \delta_0 n_k, \quad b_{0k} = ((1/3) \alpha_0 - \gamma_0) m_k - (\beta_0 + (1/3) \delta_0) n_k \quad (6.3)$$

and

$$a_{0k} + b_{0k} = (1/3) \alpha_0 m_k - \beta_0 n_k. \quad (6.4)$$

Comparing equations (1.3) and (6.1) and solving, we get

$$4 a_{0k} = -b_{0i} (\delta_k^i + 2m^i m_k). \quad (6.5)$$

Hence, we have

Theorem (6.1). In a 3-dimensional Finsler space F^3 , second order tensors a_{0k} and b_{0k} satisfy (6.4) and (6.5).

In any three dimensional Finsler space F^3 the T-tensor is defined by (6.1), which helps us give the following definition :

Definition (6.1). A Finsler space F^n ($n > 3$), shall be called T3-like Finsler space, if for arbitrary second order tensors a_{hk} and b_{hk} satisfying $a_{h0} = 0$, $b_{h0} = 0$, its T-tensor T_{ijkh} is non-zero and is expressed by an equation of the form :

$$T_{ijkh} = \Sigma_{(i,j,k)} \{a_{hk} h_{ij} + b_{hk} C_i C_j\}. \quad (6.6)$$

It is known, Shimada [13], that for second curvature tensor P_{hijk} , the Ricci tensors defined by

$$P_{hk}^{(1)} = P_{hjk}^j = C_{klh} - C_{hklj}^j + P_{kr}^j C_{jh}^r - P_{hk}^r C_r \quad (6.7)$$

and

$$P_{hk}^{(2)} = P_{hkj}^j = C_{klh} - C_{hklj}^j + C_{kh}^r C_{rl0} - P_{hr}^j C_{kj}^r \quad (6.8)$$

are non-symmetric such that

$$P_{h0}^{(1)} = 0, \quad P_{h0}^{(2)} = 0, \quad P_{0k}^{(1)} = C_{k10} = P_k, \quad P_{0k}^{(2)} = C_{k10} = P_k. \quad (6.9)$$

If we assume that the tensor $a_{hk} = P_{hk}^{(1)}$, we can obtain by virtue of equation (6.7) and $*T_{kh} = T_{ijkh} g^{ij}$

$$*T_{kh} = (n+1) P_{hk}^{(1)} + C^2 b_{hk} + 2 b_{hi} C^i C_k. \quad (6.10)$$

Since $*T_{kh}$ is symmetric in k and h , from equation (6.10), we can obtain

$$\zeta_{(k,h)} \{C_{kjh} + P_{kr}^j C_{jh}^r - (n+1)^{-1} b_{ki} (C^2 \delta_h^i + 2 C^i C_h)\} = 0, \quad (6.11)$$

therefore, we can have

Theorem (6.2). In a T3-like Finsler space F_n ($n > 3$), if tensor $a_{hk} = P_{hk}^{(1)}$, equation (6.11) is satisfied.

Multiplying equation (6.11) by h^k , we get

$$L(b_{0h} C^2 + 2b_{0i} C^i C_h) + (n+1) P_h = 0, \quad (6.12)$$

and
$$b_{0i} C^i = - \{(n+1)/3\} L^{-1} C^{-2} P_i C^i. \quad (6.13)$$

From equations (6.12) and (6.13), we can obtain

$$b_{0h} = (n+1) P_i L^{-1} C^{-2} \{(2/3) C^i C_h - \delta_h^i\}. \quad (6.14)$$

Hence, we have

Theorem (6.3). In a T3-like Finsler space F^n ($n > 3$), if the tensor $a_{hk} = P_{hk}^{(1)}$, the tensor b_{0h} is given by equation (6.14).

From equation (6.10), we can obtain on simplification

$$b_{hi} C^i = [L C_{ih} C^i + C^2 l_h - (n+1) P_{hi}^{(1)} C^i] / 3C^2. \quad (6.15)$$

Substituting the value of $b_{hi} C^i$, from equation (6.15) in (6.10), we obtain on simplification

$$b_{hk} = C^{-2} [(L C_{ih} - (n+1) P_{hi}^{(1)}) (\delta_k^i - (2/3C^2) C^i C_k) + l_k C_h + (1/3) l_h C_k]. \quad (6.16)$$

Hence, we have

Theorem (6.4). In a T3-like Finsler space F^n ($n > 3$), if the tensor a_{hk} is given by the Ricci tensor $P_{hk}^{(1)}$, the tensor b_{hk} is given by equation (6.16).

7. C-Reducible Finsler Space

In a C-reducible Finsler space F^3 , it is known that Matsumoto [4], $C_{(1)} = (3/4) C$, $C_{(2)} = 0$ and $C_{(3)} = (1/4) C$, therefore from equations (2.1) and (2.2), we can obtain

$${}^{\prime}C_{ij} = (C/4) (3 m_i m_j + n_i n_j) \quad (7.1)$$

and
$${}^*C_{ij} = (C/4) (m_i n_j + m_j n_i), \quad (7.2)$$

which lead to

$${}^{\prime}C_{ij} m^j = (3/4) C m_i, \quad {}^{\prime}C_{ij} n^j = (1/4) C n_i = {}^*C_{ij} m^j, \quad {}^*C_{ij} n^j = (1/4) C m_i. \quad (7.3)$$

Furthermore, we can obtain

$$\begin{aligned} {}^{\prime}C_{ij|_h} = (1/4) C l_h (3m_i m_j + n_i n_j) + L^{-1}(1/4) C \{ & 3\{m_i (-l_j m_h + n_j v_h) \\ & + m_j (-l_i m_h + n_i v_h)\} - \{n_i (l_j n_h + m_j v_h) + n_j (l_i n_h + m_i v_h)\} \} \end{aligned} \quad (7.4)$$

and

$$\begin{aligned} {}^*C_{ij|_h} = (1/4) [C l_h (m_i n_j + m_j n_i) - L^{-1}C \{ & m_i (l_j n_h + m_j v_h) \\ & + m_j (l_i n_h + m_i v_h) + n_i (l_j m_h - n_j v_h) + n_j (l_i m_h - n_i v_h) \}]. \end{aligned} \quad (7.5)$$

From equations (3.1) and (3.2), we can obtain

$$\begin{aligned} C_{ijk|_h} = (1/4) C l_h \{ & 3m_i m_j m_k + (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) \\ & - L^{-1} \Sigma_{(i,j,k)} (1/4) C [3(l_i m_h - n_i v_h) m_j m_k \\ & + \{l_i (m_h n_j n_k + m_j n_k n_h + m_k n_h n_j) + 2m_i m_j v_h n_k\}], \end{aligned} \quad (7.6)$$

which by virtue of symmetry of $C_{ijk|_h}$ in k and h , easily leads to

$$\begin{aligned} \zeta_{(k,h)} [& \{C_{(1)|_h} m_i m_j m_k + C_{(3)|_h} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) \} \\ & + L^{-1} [C_{(1)} \{ (m_i n_j + m_j n_i) m_k v_h + m_i m_j (l_h m_k + n_k v_h) \} \\ & + C_{(3)} \{ (m_i n_j + m_j n_i) (n_k l_h + 2v_k m_h) + (m_i m_j - n_i n_j) 2v_k n_h \} \end{aligned}$$

$$+ n_i n_j (l_h m_k + v_h n_k)] = 0. \quad (7.7)$$

Equations (3.4) to (3.11) for a C-reducible Finsler space reduce to

$$\begin{aligned} \alpha_h &= (3/4) (LCl_h + Cl_h), & \beta_h &= (1/4) Cv_h, \\ \alpha_0 &= \alpha_h l^h = (3/4) (LCl_0 + C), & \beta_0 &= \beta_h l^h = 0, & \alpha_h m^h &= (3/4) L Cl_h m^h \\ \alpha_h n^h &= (3/4) L Cl_h n^h, & \beta_h m^h &= (1/4) C v_{2)32}, & \beta_h n^h &= (1/4) C v_{2)33} \\ 3LCl_h n^h &+ Cv_{2)32} &= 0. \end{aligned} \quad (7.8)$$

For a C-reducible Finsler space equations (3.12), (3.13) and (3.14) can be expressed as

$$3C(v_h - v_{2)33} n_h) = LCl_k n^k m_h, \quad (7.9)$$

$$L Cl_h n^h - 3C v_{2)32} = 0 \quad (7.10)$$

and

$$L Cl_h m^h = C v_{2)33}. \quad (7.11)$$

From equations (7.8) and (7.10), we can obtain

$$Cl_h n^h = 0 \quad \text{or} \quad v_{2)32} = 0. \quad (7.12)$$

Hence, we have

Theorem (7.1). In a 3-dimensional C-reducible Finsler space $LCl_h m^h = Cv_{2)33}$ and $Cl_h n^h = 0$.

From equation (4.2) for a C-reducible Finsler space F^3 , we get

$$T_{ijkh} = (1/4) \Sigma_{(i,j,k)} [(LCl_h + Cl_h)(m_i m_j m_k + m_i n_j n_k) + Cv_h m_i m_j n_k], \quad (7.13)$$

which leads to

$$T_{ij} = C v_{2)33} (m_i m_j + (1/4) n_i n_j), \quad (7.14)$$

$$T_{ij} m_i = C v_{2)33} m_j, \quad T_{ij} n^i = (1/4) C v_{2)33} n_j \quad (7.15)$$

and

$$T = (5/4) C v_{2)33}. \quad (7.16)$$

Hence, we have

Theorem (7.2). In a 3-dimensional C-reducible Finsler space T_{ijk} is given by (7.13) and $T = (5/4) C_{(2)3}$.

8. P-Reducible Finsler Spaces

A Finsler space is defined as P-reducible by Matsumoto and Shimada [7] and studied by Rastogi and Kawaguchi [9] and Rastogi [12], if its torsion tensor $P_{ijk} = A_{ijk}^l 0$, is expressible as

$$P_{ijk} = (n + 1)^{-1} (A_{k10} h_{ij} + A_{i10} h_{jk} + A_{j10} h_{ki}). \quad (8.1)$$

In three dimensional Finsler space P_{ijk} on simplification can be expressed as

$$\begin{aligned} P_{ijk} = L[(C_{(1)10} + 3C_{(2)} h_0) m_i m_j m_k + (1/4) C_{10} (m_i n_j n_k + m_j n_k n_i \\ + m_k n_i n_j) + (1/4) Ch_0 (3n_i n_j n_k + m_i m_j n_k + m_j m_k n_i + m_k m_i n_j)]. \end{aligned} \quad (8.2)$$

Multiplying (8.2) by g^{jk} and using $P_i = LC_{i10}$ and $C_{10} = C_{i10} m^i$, we get

$$3 C_{10} = 4C_{(1)10} + 12 C_{(2)} h_0 \quad \text{or} \quad 4C_{(3)10} - C_{10} = 12 C_{(2)} h_0 \quad (8.3)$$

Hence, we have

Theorem (8.1). In a P-reducible Finsler space F^3 , coefficients $C_{(1)}$, $C_{(2)}$ and $C_{(3)}$ are related by equation (8.3).

From equation (2.1), we can easily obtain

$$\begin{aligned} L 'C_{ijl0} = (1/4) [(3m_i m_j + n_i n_j) \{C_{(1)10} + 3C_{(2)} h_0 + (1/4) C_{10}\} \\ + Ch_0 (m_j n_i + m_i n_j)]. \end{aligned} \quad (8.4)$$

From equation (8.4), we can obtain

$$\begin{aligned} 'C_{ijl0}^l j = 0, \quad L 'C_{ijl0} m^j = (1/4) (3 C_{10} m_i + C h_0 n_i), \\ L 'C_{ijl0} n^j = (1/4) (C_{10} n_i + C h_0 m_i). \end{aligned} \quad (8.5)$$

Similarly from equation (2.2), we can obtain

$$L *C_{ijl0} = C_{10} (m_i n_j + m_j n_i) + 4C h_0 m_i m_j - L h_0 'C_{ij}, \quad (8.6)$$

which implies

$$\begin{aligned} *C_{ijl_0} \dot{p}^j &= 0, \quad L *C_{ijl_0} m^j = m_i (4C - C_{(1)} h_0 + n_i (C_{l_0} - C_{(2)} h_0), \\ L *C_{ijl_0} n^j &= (C_{l_0} + Lh_0 C_{(2)}) m_i - L h_0 C_{(3)} n_i. \end{aligned} \quad (8.7)$$

Hence, we have

Theorem (8.2). In a P-reducible Finsler space F^3 , tensors $'C_{ij}$ and $*C_{ij}$ satisfy equations (8.6) and (8.7).

From equation (1.3) with the help of equations (8.1), we can obtain on simplification

$$\begin{aligned} LT_{ijkhl_0} &= L^2 C_{ijk|hl_0} + (n+1)^{-1} \{A_{il_0} \Sigma_{(j, k, h)} (l_j h_{kh}) + A_{jl_0} \Sigma_{(i, k, h)} (l_i h_{kh}) \\ &\quad + A_{kl_0} \Sigma_{(i, j, h)} (l_i h_{jh}) + A_{hl_0} \Sigma_{(i, j, k)} (l_i h_{jk})\}, \end{aligned} \quad (8.8)$$

which gives

Theorem (8.3). In a P-reducible Finsler space F^3 , T-tensor satisfies equation (8.8).

In a T3-like Finsler space equation (8.8) gives on simplification

$$\begin{aligned} L\Sigma_{(i, j, k)} \{a_{hkl_0} h_{ij} + b_{hkl_0} C_i C_j\} - L^2 C_{ijk|hl_0} - (n+1)^{-1} [A_{il_0} \{\Sigma_{(j, k, h)} (l_j h_{kh}) \\ - (n+1)(b_{hk} C_j + b_{hj} C_k)\} + A_{jl_0} \{\Sigma_{(i, k, h)} (l_i h_{kh}) - (n+1)(b_{hk} C_i + b_{hi} C_k)\} \\ + A_{kl_0} \{\Sigma_{(i, j, h)} (l_i h_{jh}) - (n+1)(b_{hi} C_j + b_{hj} C_i)\} + A_{hl_0} (l_i h_{jk})] = 0. \end{aligned} \quad (8.9)$$

Hence, we have

Theorem (8.4). If a T3-like Finsler space is also a P-reducible Finsler space it satisfies (8.9).

From equation (8.9), we can obtain

$$\begin{aligned} 3 a_{hkl_0} + b_{hl_0} (C^2 \delta_k^i + 2C^i C_k) - L C_k|_{hl_0} - l_k C_{hl_0} - l_h C_{kl_0} \\ = -2(n+1)^{-1} C_{il_0} (b_{hk} C^i + b_h^i C_k - b_{hj} C^j \delta_k^i). \end{aligned} \quad (8.10)$$

Hence, we have

Theorem (8.5). In a T3-like P-reducible Finsler space arbitrary tensors a_{hk} and b_{hk} are related by equation (8.10).

References

1. Hashiguchi, M. : On conformal transformations of Finsler metrics, J. Math. Kyoto Univ., 16 (1976), 25-50.
2. Izumi, H. : On P*-Finsler space I, Memo. Defence Academy, 16 (1976), 133-138.
3. Kawaguchi, H. : On the Finsler spaces with the vanishing second curvature tensor, Tensor, N. S. 26 (1972), 250-254.
4. Matsumoto, M. : On C-reducible Finsler space, Tensor, N. S., 24 (1972), 29-37.
5. Matsumoto, M. : V-transformation of Finsler spaces. I. Definitions, infinitesimal transformations and isometries, J. Math. Kyoto Univ., 12 (1972), 479-512.
6. Matsumoto, M. : Foundations of Finsler Geometry and special Finsler spaces, Kaiseisha Press, Otsu, 520, Japan, 1986.
7. Matsumoto, M. and Shimada, H. : On Finsler spaces with the curvature tensor P_{hijk} and S_{hijk} satisfying special conditions, Rep. on Math. Phys., 12 (1977), 77-87.
8. Matsumoto, M. and Shimada, H. : On Finsler spaces with 1-form metric, Tensor, N. S., 32 (1978), 161-169.
9. Rastogi, S. C. and Kawaguchi, H. : A geometrical meaning of the P-reducible condition in Finsler spaces, Tensor, N. S., 51 (1992), 251-256.
10. Rastogi, S. C. : On some new tensors and their properties in a Finsler space, J.T.S.I., 8, 9, 10 (1990-92), 12-21.
11. Rastogi, S. C. : T2 and TR2-like Finsler spaces, J. Nat. Acad. Math., 17 (2003), 1-8.
12. Rastogi, S. C. : On certain P-reducible Finsler spaces, Ganita, 56, 1 (2005), 55-64.
13. Shimada, H. : On the Ricci tensors of particular Finsler spaces, J. Korean Math. Soc., 14, 1 (1977), 41-63.

