Hypersurface of a Special Finsler Space with metric $\alpha + \beta + \frac{\alpha^2}{(\alpha - \beta)}$

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Abstract

In the present paper our study confines to the hypersurface of a Finsler space with $(\alpha, \beta)$ metric $\alpha + \beta + \frac{\alpha^2}{(\alpha - \beta)}$. We have examined the hypersurfaces as a hyperplane of first, second and third kinds.

1. Introduction

We consider an n-dimensional Finsler space $F^n = (M^n, L)$ i.e., a pair consisting of an n-dimensional differentiable manifold $M^n$ equipped with a Fundamental function $L$. The concept of $(\alpha, \beta)$, metric $L(\alpha, \beta)$ was introduced first of all by M. Matsumoto [5] and has been studied by many authors [1, 2, 3, 5, 8, 7, 9]. As well known examples are Randers metric $(\alpha + \beta)$, Kropina metric $\frac{a^2}{\beta}$ and generalized Kropina metric $\frac{m+1}{\beta}$ ($m \neq 0, -1$) whose studies have greatly contributed a lot to the growth of Finsler geometry. A Finsler metric $L(x, y)$ is called an $(\alpha, \beta)$ metric if $L$ is a positively homogeneous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M^n$.

2. Preliminaries

We devote to a special Finsler Space $F^n = \{M^n, L(\alpha, \beta)\}$ with the metric

\begin{equation}
L(\alpha, \beta) = \alpha + \beta + \frac{\alpha^2}{(\alpha - \beta)}
\end{equation}

Partial derivative of (2.1) w.r.t $\alpha$ and $\beta$ are given by

$$L_\alpha = \frac{2\alpha^2 + \beta^2 - 4\alpha\beta}{(\alpha - \beta)^2}, \quad L_\beta = \frac{2\alpha^2 + \beta^2 - 2\alpha\beta}{(\alpha - \beta)^2}$$
\[ L_{aa} = \frac{2\beta^2}{(\alpha - \beta)^3}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha - \beta)^3}, \quad L_{\alpha\beta} = -\frac{2\alpha\beta}{(\alpha - \beta)^3} \]

where \( L_{\alpha} = \frac{\partial L}{\partial \alpha}, \quad L_{\beta} = \frac{\partial L}{\partial \beta}, \quad L_{aa} = \frac{\partial L_{\alpha}}{\partial \alpha}, \quad L_{\beta\beta} = \frac{\partial L_{\beta}}{\partial \beta}, \quad L_{\alpha\beta} = \frac{\partial L_{\alpha}}{\partial \beta}. \)

In the Finsler space \( F^n = \{ M^n, L(\alpha, \beta) \} \) the normalized element of support \( l_i = \partial_i L \) and angular metric tensor \( h_{ij} \) are given by [5]:

\[ l_i = \alpha^{-1} L_i Y_i + L_\beta b_i \]
\[ h_{ij} = p_{aij} + q_{0} b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2} Y_i Y_j \]

where \( Y_i = a_{ij} y^j \). For the fundamental function (2.1) above constants are

\[ p = LL_{\alpha} \alpha^{-1} = \frac{4\alpha^4 - \beta^4 - 8\alpha^3 \beta + 4\alpha \beta^3}{\alpha(\alpha - \beta)^3} \]
\[ q_0 = LL_{\beta\beta} = \frac{4\alpha^4 - 2\alpha^2 \beta^2}{(\alpha - \beta)^4}, \quad q_{-1} = LL_{\alpha \beta} \alpha^{-1} = \frac{2\beta^3 - 4\alpha^2 \beta}{(\alpha - \beta)^4} \]
\[ q_{-2} = L(\alpha a_{\alpha a} - L_{\alpha} \alpha^{-1}) = \frac{-4\alpha^5 - 2\alpha^2 \beta^3 + 8\alpha^4 \beta + \alpha \beta^4 - \beta^5}{\alpha^3(\alpha - \beta)^4} \]

Fundamental metric tensor \( g_{ij} = \frac{1}{2} \partial_i \partial_j L^2 \) and its reciprocal tensor \( g^{ij} \) for \( L = L(\alpha, \beta) \) are given by [4, 5]

\[ g_{ij} = p_{aij} + p_{0} b_i b_j + p_{-1}(b_i Y_j + b_j Y_i) + p_{-2} Y_i Y_j \]

where

\[ p_{0} = q_0 + L_{\beta}, \quad p_{-1} = q_{-1} + L^{-1} p L_{\beta} = \frac{2\alpha \beta^3 - 4\alpha^3 \beta + (2\alpha^2 + \beta^2 - 2\alpha \beta)^2}{\alpha(\alpha - \beta)^4} \]
\[ p_{-2} = q_{-2} + p^2 L^{-2} = \frac{2\beta^4 + 8\alpha^2 \beta^2 - 6\alpha^3 \beta + \beta^5}{\alpha^2(\alpha - \beta)^4} \]

The reciprocal tensor \( g^{ij} \) of \( g_{ij} \) is given by

\[ g^{ij} = p^{-1} a^{ij} - s_0 b_i b_j - s_{-1}(b^i y^j + b^j y^i) - s_{-2} y^i y^j \]

where \( b^i = a^{ij} b_j \) and \( b^2 = a_{ij} b^i b^j \).
Hypersurface of a Special Finsler Space with metric . . .

\( s_0 = \frac{1}{\tau^p} \{p p_0 + (p_0 p - p_2^2 - p_1^2) \alpha^2 \}, \)

\( s_{-1} = \frac{1}{\tau^p} \{p p_{-1} + (p_0 p - p_2^2) \beta \}, \)

\( s_{-2} = \frac{1}{\tau^p} \{p p_{-2} + (p_0 p - p_2^2) b^2 \}, \)

\( \tau = \rho (p + p_0 b^2 + p_{-1} \beta) + (p_0 p_{-2} - p_2^2)(\alpha^2 b^2 - \beta^2) \)

The hv-torsion tensor \( C_{ijk} = \frac{1}{2} \partial_k g_{ij} \) is given by \([10]\)

\( 2p C_{ijk} = p_{-1} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \gamma_1 m_i m_j m_k \)

where,

\( \gamma_1 = p \partial p_0 \partial \beta - 3p_{-1} q_{00}, \quad m_i = b_i - \alpha^{-2} \beta Y_i \)

Here \( m_i \) is a non-vanishing covariant vector orthogonal to the element of support \( y^i \).

Let \( \{ \frac{i}{j} \} \) be the component of christoffel symbols of the associated Riemannian space \( R^n \) and \( \nabla_k \) be the covariant derivative with respect to \( x^k \) relative to this christoffel symbol. Now we define,

\( 2E_{ij} = b_{ij} + b_{ji}, \quad 2F_{ij} = b_{ij} - b_{ji} \)

where \( b_{ij} = \nabla_j b_i \).

Let \( CT = (\Gamma^i_{jk}, \Gamma^i_{0k}, \Gamma^i_{jk}) \) be the cartan connection of \( F^n \). The difference tensor \( D^i_{jk} = \Gamma^i_{jk} - \{ \frac{i}{j} \} \) of the special Finsler space \( F^n \) is given by

\( D^i_{jk} = B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_k b_{0k} + B^i_k b_{0j} \)

\( \quad -b_0 m g^i m B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_k g^{is} \)

\( \quad + \lambda^2 (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^i_{jkm} C^m_{ms}) \)

where
\[(2.11)\]

\[B_k = p_0 b_k + p_{-1} Y_k, \quad B^i = g^{ij} B_j, \quad F^k_i = g^{kj} F_{ji}\]

\[B_{ij} = \left\{ p_{-1}(\alpha_{ij} - \alpha^{-2} Y_i Y_j) + \frac{\partial \alpha}{\partial y^i} m_i m_j \right\} / 2, \quad B^k_i = g^{kj} B_{ji}\]

\[A^k_m = B^m_k E_0 + B^m_k E_{k0} + B_k F^m_0 + B_0 F^m_k\]

\[\lambda^m = B^m E_0 + 2 B_0 F^m_0, \quad B_0 = B_i y^i\]

where ‘0’ denote contraction with \(y^i\) except for the quantities \(p_0, q_0\) and \(s_0\).

3. Induced Cartan Connection

Let \(F^{n-1}\) be a hypersurface of \(F^n\) given by the equation \(x^i = x^i(u^\alpha)\) (where \(\alpha = 1, 2, 3, \ldots (n-1)\)). The element of support \(y^i\) of \(F^n\) is to be taken tangential to \(F^{n-1}\), that is [6],

\[(3.1)\]

\[y^i = B^i_\alpha(u) u^\alpha\]

the metric tensor \(g_{\alpha\beta}\) and hv-tensor \(C_{\alpha\beta\gamma}\) of \(F^{n-1}\) are given by

\[g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta, \quad C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma\]

and at each point \((u^\alpha)\) of \(F^{n-1}\), a unit normal vector \(N^i(u, v)\) is defined by

\[g_{ij} \{x(u, v), y(u, v)\} B^0_i N^j = 0, \quad g_{ij} \{x(u, v), g(u, v)\} N^i N^j = 1\]

Angular metric tensor \(h_{\alpha\beta}\) of the hypersurface are given by

\[(3.2)\]

\[h_{\alpha\beta} = h_{ij} B^i_\alpha B^j_\beta, \quad h_{ij} B^i_\alpha N^j = 0, \quad h_{ij} N^i N^j = 1\]

\((B^i_\alpha, N_i)\) inverse of \((B^i_\alpha, N^i)\) is given by

\[B^i_\alpha = g^{ij} g_{j\beta} B^j_\beta, \quad B^0_i B^i_\beta = \delta_\alpha^\beta, \quad B^0_i N^i = 0, \quad B^i_\alpha N_i = 0\]

\[N_i = g_{ij} N^j, \quad B^k_i = g^{kj} B_{ji}, \quad B^0_i B^i_\beta + N^i N_j = \delta^0_j\]

The induced connection \(IC = (\Gamma^i_{\alpha\beta}, G^i_{\alpha\beta}, C^i_{\alpha\beta\gamma})\) of \(F^{n-1}\) from the Cartan’s connection \(C = (\Gamma^i_{jk}, \Gamma^i_{0k}, C^i_{jk})\) is given by [6],

\[\Gamma^i_{\alpha\beta} = B^i_\alpha (B^j_{\beta\gamma} + \Gamma^i_{jk} B^j_{\beta\gamma}) + M^0_{\alpha\beta} H_\gamma\]

\[G^i_{\alpha\beta} = B^i_\alpha (B^j_{0\beta} + \Gamma^i_{0j} B^j_{\beta\gamma}), \quad C^i_{\alpha\beta\gamma} = B^0_i C^j_{ik} B^j_{\beta\gamma}\]

where

\[M_{\beta\gamma} = N_i C^i_{jk} B^j_{\beta\gamma}, \quad M^0_{\alpha\beta} = g^{\alpha\gamma} M_{\beta\gamma}, \quad H_\beta = N_i (B^i_{0\beta} + \Gamma^i_{0j} B^j_{\beta\gamma})\]

and

\[B^i_{\beta\gamma} = \frac{\partial B^i_{\beta}}{\partial u^\gamma}, \quad B^i_{0\beta} = B^i_{\alpha\beta} u^\alpha\]
The quantities $M_{\beta\gamma}$ and $H_\beta$ are called the second fundamental v-tensor and normal curvature vector respectively [6]. The second fundamental h-tensor $H_{\beta\gamma}$ is defined as [6]

$$H_{\beta\gamma} = N_i (B^i_{\beta\gamma} + \Gamma^i_{jk} B^j_{\beta} B^k_{\gamma}) + M_\gamma H_\gamma$$

where

$$M_\beta = N_i C^i_{jk} B^j_{\beta} N^k$$

The relative h and v-covariant derivatives of projection factor $B^i_\alpha$ with respect to ICT are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \quad B^i_{\alpha|\beta} = M_{\alpha\beta} N^i$$

It is obvious form the equation (3.3) that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta$$

The above equation yield

$$H_{\beta0} = H_\gamma, \quad H_{\gamma0} = H_\gamma + M_\gamma H_0$$

We shall use following lemmas which are due to Matsumoto [6] in the coming section

**Lemma 3.1.** The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector $H_\beta$ vanishes.

**Lemma 3.2.** A hypersurface $F^{n-1}$ is a hyperplane of the first kind with respect to connection CT if and only if $H_\alpha = 0$.

**Lemma 3.3.** A hypersurface $F^{n-1}$ is a hyperplane of the second kind with respect to connection CT if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

**Lemma 3.4.** A hypersurface $F^{n-1}$ is a hyperplane of the third kind with respect to connection CT if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4. **Hypersurface $F^{n-1}(c)$ of a special Finsler space**

Let us consider a Finsler space with the metric $L = \alpha + \beta + \frac{\alpha^2}{\alpha - \beta}$, where, vector field $b_i(x) = \frac{\partial b}{\partial x^i}$ is a gradient of some scalar function $b(x)$. Now we consider a hypersurface $F^{n-1}(c)$ given by equation $b(x) = c$, a constant [10].

From the parametric equation $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get
\[
\frac{\partial b(x)}{\partial u^\alpha} = 0 \\
\frac{\partial b(x)}{\partial x^i} \frac{\partial x^i}{\partial u^\alpha} = 0 \\
b_i B_\alpha^i = 0
\]

Above shows that \(b_i(x)\) are covariant component of a normal vector field of hypersurface \(F^{n-1}(c)\). Further, we have

\[(4.1) \quad b_i B_\alpha^i = 0 \quad \text{and} \quad b_i y^j = 0 \quad \text{i.e.} \quad \beta = 0\]

and induced matric \(L(u, v)\) of \(F^{n-1}(c)\) is given by

\[(4.2) \quad L(u, v) = a_{\alpha \beta} u^\alpha v^\beta, a_{\alpha \beta} = a_{ij} B_\alpha^i B_\beta^j\]

which is a Riemannian metric.

Writing \(\beta = 0\) in the equations (2.2), (2.3) and (2.5) we get

\[(4.3) \quad p = 4, \quad q_0 = 4, \quad q_{-1} = 0, \quad q_{-2} = -4\alpha^{-2} \]

\[p_0 = 8, \quad p_{-1} = 4\alpha^{-1}, \quad p_{-2} = 0, \quad \tau = 16(1 + b^2),\]

\[s_0 = \frac{1}{4(1 + b^2)}, \quad s_{-1} = \frac{1}{4\alpha(1 + b^2)}, \quad s_{-2} = \frac{-b^2}{4\alpha^2(1 + b^2)}\]

from (2.4) we get,

\[(4.4) \quad g^{ij} = a^{ij} - \frac{1}{4(1 + b^2)} b_i b_j - \frac{1}{4\alpha(1 + b^2)} (b_i y^j + b_j y^i) + \frac{b^2}{4\alpha^2(1 + b^2)} y^i y^j\]

thus along \(F^{n-1}(c)\), (4.4) and (4.1) leads to

\[g^{ij} b_i b_j = \frac{b^2}{4(1 + b^2)}\]

So we get

\[(4.5) \quad b_i(x(u)) = \sqrt{\frac{b^2}{4(1 + b^2)}} N_i, \quad b^2 = a^{ij} b_i b_j\]

where \(b\) is the length of the vector \(b^i\).

Again from (4.4) and (4.5), we get

\[(4.6) \quad b^i = a^{ij} b_j = \sqrt{\frac{4b^2(1 + b^2)}{(1 + b^2(1 - \alpha^2))}} N^i + \frac{\alpha b^2 y^i}{1 + b^2(1 - \alpha^2)}\]

thus we have,
Theorem 4.1. In a special Finsler hypersurface $F^{n-1}(c)$, the Induced Riemannian metric is given by (4.2) and the scalar function $b(x)$ is given by (4.5) and (4.6).

Now the angular metric tensor $h_{ij}$ and metric tensor $g_{ij}$ of $F^n$ are given by

$$
(4.7) \quad h_{ij} = 4a_{ij} + 4b_i b_j - \frac{4}{\alpha^2} Y_i Y_j \quad \text{and} \quad g_{ij} = 4a_{ij} + 8b_i b_j + \frac{4}{\alpha} (b_i Y_j + b_j Y_i)
$$

From equation (4.1), (4.7) and (3.2) it follows that if $h^\alpha_{\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$ then we have along $F^{n-1}_c$, $h_{\alpha\beta} = h^\alpha_{\beta}$.

Thus along $F^{n-1}_c$, \[ \frac{\partial p_0}{\partial \beta} = \frac{24}{\alpha}. \]

From equation (2.6) we get

$$
Y_1 = \frac{48}{\pi}, \quad m_i = b_i
$$

then hv-torsion tensor becomes

$$
(4.8) \quad C_{ijk} = \frac{1}{2\alpha} (h_{ij} b_k + h_{jk} b_i + h_{ki} b_j) + \frac{6}{\alpha} b_i b_j b_k
$$

in the special Finsler hypersurface $F^{n-1}_c$. Due to fact from (3.2), (3.3), (3.5), (4.1) and (4.8) we have

$$
(4.9) \quad M_{\alpha\beta} = \frac{1}{2\alpha} \sqrt{\frac{b^2}{4(1+b^2)}} h_{\alpha\beta} \quad \text{and} \quad M_\alpha = 0
$$

Therefore from equation (3.6) it follows that $H_{\alpha\beta}$ is symmetric. Thus we have

Theorem 4.2. The second fundamental v-tensor of the special Finsler hypersurface $F^{n-1}_c$ is given by (4.9) and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Now from (4.1) we have $b_i B^i_\alpha = 0$. Then we have

$$
(4.10) \quad b_{ij|\beta} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0
$$

Therefore, from (3.5) and using $b_{ij|\beta} = b_{ij} B^j_\beta + b_i |j N^j H_\beta$, we have

$$
(4.11) \quad b_{ij} B^i_\alpha B^j_\beta + b_{ij} B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0
$$

since $b_i |j = -b_i C^h_{ij}$, we get

$$
(4.12) \quad b_{ij} B^i_\alpha N^j = 0
$$
Therefore from equation (4.10) we have,

\[ \sqrt{\frac{b^2}{4(1 + b^2)}} H_\alpha + b_{ij} B^i_\alpha B^j_\beta = 0 \]

because \( b_{ij} \) is symmetric. Now contracting (4.11) with \( v^\beta \) and using (3.1) we get

\[ \sqrt{\frac{b^2}{4(1 + b^2)}} H_\alpha + b_{ij} B^i_\alpha y^j = 0 \]

Again contracting by \( v^\alpha \) equation (4.12) and using (3.1), we have

\[ \sqrt{\frac{b^2}{4(1 + b^2)}} H_0 + b_{ij} y^i y^j = 0 \]

From lemma (3.1) and (3.2), it is clear that the hypersurface \( F_{(c)}^{n-1} \) is a hyperplane of first kind if and only if \( H_0 = 0 \). Thus from (4.13) it is obvious that \( F_{(c)}^{n-1} \) is a hyperplane of first kind if and only if \( b_{ij} y^i y^j = 0 \). This \( b_{ij} \) being the covariant derivative with respect to \( CT \) of \( F^n \) defined on \( y^i \), but \( b_{ij} = \nabla_j b_i \) is the covariant derivative with respect to Riemannian connection \( \{ \Gamma_{jk}^i \} \) constructed from \( a_{ij}(x) \).

Hence \( b_{ij} \) does not depend on \( y^i \). We shall consider the difference \( b_{ij} - b_{ij} \) where \( b_{ij} = \nabla_j b_i \) in the following. The difference tensor \( D^i_{jk} = \Gamma_{jk}^{si} - \{ \Gamma_{jk}^i \} \) is given by (2.10). Since \( b_i \) is a gradient vector, then from (2.9) we have

\[ E_{ij} = b_{ij} \quad F_{ij} = 0 \quad and \quad F^i_j = 0. \]

Thus (2.10) reduces to

\[ D^i_{jk} = B^i_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m s g^{is} + \lambda^s (C^i_{jm} C^m_{sk} + C^m_{kj} C^m_{sj} - C^m_{jkm}) \]

where
(4.15) \[ B_i = 8b_i + 4\alpha^{-1}Y_i, \quad B^i = \left( \frac{1}{1 + b^2} \right)b^i + \frac{1}{\alpha(1 + b^2)}y^i, \]

\[ \lambda^m = B^mb_{00}, \quad B_{ij} = \frac{2}{\alpha}(a_{ij} - \frac{Y_i Y_j}{\alpha^2}) + \frac{12}{\alpha}b_ib_j, \]

\[ B^i_j = \frac{1}{2\alpha}(\delta^i_j - \alpha^{-1}y^i Y_j) + \frac{5}{2\alpha(1 + b^2)}b^i b_j - \frac{(1 + 6b^2)}{2\alpha^2(1 + b^2)}b_j Y^i, \quad A^m_k = B^m_k b_{00} + B^m_k b_{00}. \]

In view of (4.3) and (4.4), the relation in (2.11) becomes to by virtue of (4.15) we have \( B_{00} = 0, B_{i0} = 0 \) which leads \( A^m_0 = B^m b_{00} \).

Now contracting (4.14) by \( y_k \) we get

\[ D^i_{j0} = B^i b_{j0} + B^i b_{00} - B^m C^i_{jm} b_{00} \]

Again contracting the above equation with respect to \( y^j \) we have

\[ D^i_{00} = B^i b_{00} = \left\{ (\frac{1}{1 + b^2})b^i + \frac{1}{\alpha(1 + b^2)}y^i \right\} b_{00} \]

Paying attention to (4.1), along \( F^{n-1}_c \), we get

(4.16) \[ b_i D^i_{j0} = \left( \frac{b^2}{1 + b^2} \right)b_{j0} + \left( \frac{1 + 6b^2}{2\alpha(1 + b^2)} \right)b_{j0} b_{00} + \frac{1}{(1 + b^2)}b_ib^m C^i_{jm} b_{00} \]

Now we contract (4.16) by \( y^i \) we have

(4.17) \[ b_i D^i_{00} = \frac{1}{(1 + b^2)}b_{00} \]

From (3.3), (4.5), (4.6), (4.9) and \( M_\alpha = 0 \), we have

\[ b_ib^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0. \]

Thus the relation \( b_{ij} = b_{ij} - b_r D^r_{ij} \) the equation (4.16) and (4.17) gives

\[ b_{ij} y^i y^j = b_{00} - b_r D^r_{00} = \frac{1}{1 + b^2}b_{00}. \]

Consequently (4.12) and (4.13) may be written as

(4.18) \[ \sqrt{\frac{b^2}{4(1 + b^2)}} H_\alpha + \frac{1}{1 + b^2}b_{00} B^i_\alpha = 0, \]

\[ \sqrt{\frac{b^2}{4(1 + b^2)}} H_0 + \frac{1}{1 + b^2}b_{00} = 0 \]
Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$. Using the fact $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i b_j$ for some $c_j(x)$. Thus we can write,

$$2b_{ij} = b_i c_j + b_j c_i$$

Now from (4.1) and (4.19) we get

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} y^i y^j = 0.$$ 

Hence from (4.18) we get $H_\alpha = 0$, again from (4.19) and (4.15) we get $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{2}{\alpha} h_{\alpha\beta}$.

Now we use equation (3.3), (4.4), (4.5), (4.6), (4.9) and (4.14) then we have

$$b_r D^r_j B^i_\alpha B^j_\beta = \frac{-c_0 b^2(4 + 3b^2)}{16\alpha(1 + b^2)^2} h_{\alpha\beta}$$

Thus the equation (4.11) reduces to

$$\sqrt{\frac{b^2}{4(1 + b^2)}} H_\alpha + \frac{b^2(4 + 3b^2)}{16\alpha(1 + b^2)^2} h_{\alpha\beta} = 0$$

Hence the hypersurface $F^{n-1}_{(c)}$ is umbilic.

**Theorem 4.3.** The necessary and sufficient condition for $F^{n-1}_{(c)}$ to be a hyperplane of first kind is (4.19). In this case the second fundamental tensor of $F^{n-1}_{(c)}$ is proportional to its angular metric tensor.

Now from lemma (3.3), $F^{n-1}_{(c)}$ is a hyperplane of second kind if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$. Thus from (4.20), we get

$$c_0 = c_i(x) y^i = 0$$

Therefore there exist a function $\psi(x)$ such that

$$c_i(x) = \psi(x) b_i(x)$$

Therefore (4.19) we get

$$2b_{ij} = b_i(x) \psi(x) b_j(x) + b_j(x) \psi(x) b_i(x)$$

This can also be written as

$$b_{ij} = \psi(x) b_i b_j$$

**Theorem 4.4.** The necessary and sufficient condition for a hypersurface $F^{n-1}_{(c)}$ to be a hyperplane of second kind is (4.21).
Again lemma (4.4), together with (4.9) and $M_\alpha = 0$ shows that $F^{n-1}_\alpha$ does not become a hyperplane of third kind.

**Theorem 4.5.** The hypersurface $F^{n-1}_\alpha$ is not a hyperplane of the third kind.

**References**


