Characterization of Quarter Symmetric Non-Metric Connection on Transversal Hypersurfaces of Lorentzian para-Sasakian Manifolds

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Abstract

In the present paper, quarter symmetric non metric connection on transversal hypersurfaces of Lorentzian para-Sasakian manifold is defined. It is studied the characterization of connections for product structure and it is shown that each transversal hypersurfaces of Lorentzian para-Sasakian manifold admits an almost product Lorentzian structure on a quarter symmetric non metric connection. Some characterization of transversal hypersurfaces of Lorentzian para-Sasakian manifold with a quarter symmetric non metric connection are studied which are closed.

Keywords and Phrases: Lorentzian almost paracontact manifolds, Almost product metric structure, Transversal hypersurfaces, Lorentzian almost paracontact Sasakian manifolds, Quarter Semi-symmetric non metric connection.

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1. Introduction

The notion of Lorentzian para contact manifold was introduced by K. Matsumoto [2]. The properties of Lorentzian para contact manifolds and their different classes, viz LP-Sasakian and LSP-Sasakian manifolds, have been studied by several authors since then. S. Tanno [8] gave a classification for connected almost contact metric manifolds whose automorphism groups have the maximum dimension. For such a manifold, the sectional curvature of plane sections containing $xi$ is a constant, say $c$. He showed that they can be divided into three classes: (1) Homogenous normal contact Riemannian manifolds with $c > 0$, (2)
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global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $R \times f^{C^n}$ if $c < 0$. It is known that the manifolds of class (1) are characterized by some tensorial relations admitting a contact structure. Kenmotsu [1] characterized the differential geometric properties of the third case by tensor equation $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$. The structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [1]. Yano and Okumura [4] introduced $(f, g, u, v, \lambda)$ structure on a manifold. Transversal hypersurfaces is a hypersurface which never contain the vector field $\xi$ defining the almost contact structure. It is well known that on a transversal hypersurface of almost contact metric manifold there exist a $(f, g, u, v, \lambda)$ structure. Let $\nabla$ be a linear connection in an $n$-dimensional differentiable manifold $M$. The torsion tensor $T$ and the curvature tensor $R$ of $\nabla$ are respectively given by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

The connection $\nabla$ is symmetric if the torsion tensor $T$ vanishes, otherwise it is non-symmetric. The connection $\nabla$ is metric if there is a Riemannian metric $g$ in $M$ such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In [6], S. Golab introduced the idea of a quarter-symmetric connection. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $T$ is of the form

$$T(X, Y) = u(Y)\phi X - u(X)\phi Y,$$

where $u$ is a 1-form and $\phi$ is a tensor field of type $(1, 1)$. Some properties of quarter symmetric connections are studied in [5, 9]. In [7], S. Rahman studied Transversal hypersurfaces of almost hyperbolic contact manifolds with a quarter symmetric non metric connection respectively.

The paper is organized as follows: In section 2, we give a brief introduction to Lorentzian para-Sasakian manifolds. In Section 3, It is proved that transversal hypersurfaces of Lorentzian para-Sasakian manifold with a quarter symmetric non metric connection admits an almost product structure and each transversal hypersurfaces of Lorentzian para-Sasakian manifold with a quarter symmetric non metric connection admits an almost product semi-Riemannian structure. The fundamental 2-form on the transversal hypersurfaces of Lorentzian para-Sasakian manifold with Lorentzian para-contact structure $(f, g, u, v, \lambda)$ structure
are closed. It is also proved that transversal hypersurfaces of Lorentzian para-Sasakian manifold with a quarter symmetric non metric connection admits a product structure. Some properties of transversal hypersurfaces of Lorentzian para-Sasakian manifold are closed.

2. Lorentzian para-Sasakian manifolds

If on an \( n \)-dimensional differentiable manifold \( M \) of differentiability class \( C^{r+1} \), there exist a vector valued linear function \( \phi \), a 1-form \( \eta \), the associated vector field \( \xi \) and the Lorentzian metric \( g \) satisfying

\[
\phi^2 = I + \eta \otimes \xi \tag{2.1}
\]
\[
\eta(\phi X) = 0 \tag{2.2}
\]
\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.3}
\]

for vector fields \( X \) and \( Y \), then \( (M, g) \) is said to be Lorentzian almost para contact manifold and the structure \( (\phi, \eta, \xi, g) \) is called Lorentzian almost para contact structure on \( M \) [2]. In view of (2.1), (2.2) and (2.3), we find

\[
\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad \phi(\xi) = 0 \tag{2.4}
\]

If moreover,

\[
(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.5}
\]
\[
\tilde{\nabla}_X \xi = \phi X \tag{2.6}
\]

where \( \tilde{\nabla} \) denotes the operator of covariant differentiation with respect to the Lorentzian metric \( g \), then \( (M, \phi, \xi, \eta, g) \) is called Lorentzian para Sasakian manifold [3], [4]. On other hand, a quarter symmetric non metric connection \( \nabla \) on \( M \) is defined by

\[
\nabla_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi X \tag{2.7}
\]

Using (2.1) and (2.6) in (2.4) and (2.5), we get respectively

\[
(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(X)\eta(Y)\xi \tag{2.8}
\]
\[
\nabla_X \xi = 0 \tag{2.9}
\]

3. Main results

Let \( M \) be a hypersurface of an almost hyperbolic contact manifold \( \tilde{M} \) equipped with an almost hyperbolic contact structure \( (\phi, \xi, \eta) \). We assume that the structure vector field \( \xi \) never belongs to tangent space of the hypersurface \( M \), such that a hypersurface is called a transversal hypersurface of an almost contact manifold. In this case the structure vector field \( \xi \) can be taken as an
affine normal to the hypersurface. Vector field $X$ on $M$ and $\xi$ are linearly independent, therefore we may write

$$\phi X = F(X) + \omega(X)\xi$$

(3.1)

where $F$ is a $(1,1)$ tensor field and $\omega$ is a 1-form on $M$. From (3.1) we have

$$\phi \xi = F\xi + \omega(\xi)\xi$$

or

$$0 = F\xi + \omega(\xi)\xi$$

$$\phi^2 X = F(\phi X) + \omega(\phi X)\xi$$

(3.2)

$$X + \eta(X)\xi = F(FX + \omega(X)\xi) + \omega(FX + \omega(X)\xi)\xi$$

$$X + \eta(X)\xi = F^2 X + (\omega \circ F)(X)\xi$$

(3.3)

Taking account of equation (3.3) we get

$$F^2 X = X$$

(3.4)

$$F^2 = I$$

(3.5)

$$\eta = \omega \circ F$$

Thus we have

**Theorem 3.1.** Each transversal hypersurface of a Lorentzian almost paracontact manifold endowed with a quarter symmetric non metric connection admits an almost product structure $F$ and a 1-form $\omega$.

From (3.4) and (3.5), it follows that

$$\eta = \omega \circ F$$

$$\eta(FX) = (\omega \circ F)FX$$

$$\eta(FX) = \omega(F^2 X)$$

$$(\omega \circ F)X = \omega(X)$$

$$\omega = \eta \circ F$$

(3.6)

Now, we assume that $\overline{M}$ admits a Lorentzian almost paracontact metric structure $(\phi, \xi, \eta, g)$ endowed with a quarter symmetric non metric connection. We denote by $g$ the induced metric on $\overline{M}$ also. Then for all $X, Y \in TM$, we obtain

$$g(FX, FY) = g(X, Y) + \eta(X)\eta(Y) + \omega(X)\omega(Y)$$

(3.7)

We define a new metric $G$ on the transversal hypersurface given by

$$G(X, Y) = g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

(3.8)
So,
\[ G(FX, FY) = g(FX, FY) + \eta(FX)\eta(FY) \]
\[ = g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y) + (\eta \circ F)(X)(\eta \circ F)(Y) \]
\[ = g(X, Y) + \eta(X)\eta(Y) - \omega(X)\omega(Y) + \omega(X)\omega(Y) \]
\[ = g(X, Y) + \eta(X)\eta(Y) = G(X, Y) \]

Then we get
\[ G(FX, FY) = G(X, Y) \]
where equations (3.4), (3.6), (3.7) and (3.8) are used. Then \( G \) is Lorentzian metric on \( M \) that is \( (F, G) \) is an almost product Lorentzian structure on the transversal hypersurface \( M \) of \( \overline{M} \).

Thus, we are able to state the following

**Theorem 3.2.** Each transversal hypersurface of Lorentzian almost paracontact manifold endowed with a quarter symmetric non metric connection admits an almost product Lorentzian structure.

We now assume that \( M \) is orientable and choose a unit vector field \( N \) of \( \overline{M} \), normal to \( M \). Then Gauss and Weingarten formulae of quarter symmetric non metric connection are given respectively by

\[ \nabla_X Y = \nabla_X Y + (h(X, Y) + u(X)u(Y))N, \quad X, Y \in TM, \quad (3.10) \]

\[ \nabla_X N = -HX + fX + u(X)\lambda N \quad (3.11) \]

where \( \nabla \) and \( \nabla \) are respectively the Levi-Civita connections in \( \overline{M} \) and \( M \), and \( h \) is the second fundamental form related to \( H \) by

\[ h(X, Y) = g(HX, Y) \quad (3.12) \]

For any vector field \( X \) tangent to \( M \), defining

\[ \phi X = fX + u(X)N \quad (3.13) \]

\[ \phi N = -U \quad (3.14) \]

\[ \xi = V + \lambda N \quad (3.15) \]

\[ \eta(X) = v(X) \]

\[ \lambda = \eta(N) = g(\xi, N) \quad (3.16) \]

for \( X \in TM \) we get an induced Lorentzian \( (f, g, u, v, \lambda) \)-structure on the transversal hypersurface such that

\[ f^2 = I + u \otimes V + v \otimes U \quad (3.17) \]

\[ fU = -\lambda V, \quad fV = \lambda U \quad (3.18) \]
\begin{equation}
\phi X = FX + \omega(X) \xi
\end{equation}
\begin{equation}
\xi = V + \lambda N
\end{equation}
\begin{equation}
\phi X = FX + \omega(X) V + \lambda \omega(X) N,
\end{equation}
\begin{equation}
\phi X = fX + u(X) N.
\end{equation}
From equation (3.30) and (3.31) we have
\begin{equation}
\lambda \omega X = u(X), \omega(X) = \frac{1}{\lambda} u(X),
\end{equation}
\[
FX = fX - \omega(X)V, \\
FX = f - \frac{1}{\lambda} u(X)V \\
F = f - \frac{1}{\lambda} u \otimes V
\]
which is equation (3.25).

\[
(uoF)(X) = (uof)(X) - \frac{1}{\lambda} u(X)u(V) \\
uoF = uof = \lambda v
\]
which gives equation (3.27).

\[
FU = fV - \frac{1}{\lambda} u(v)V \\
FU = -\lambda V - \frac{1}{\lambda}(-1 - \lambda^2)V = \frac{1}{\lambda} V \\
FU = \frac{1}{\lambda} V
\]
which gives equation (3.26)

\[
(uoF)(X) = (uof)(X) - \frac{1}{\lambda} u(X)u(V) \\
= (uof)(X) - \frac{1}{\lambda} u(X)(-1 - \lambda^2) \\
= -\lambda u(X) + \frac{1}{\lambda} u(X) + \lambda(X) \\
= \frac{1}{\lambda} u(X) \\
uoF = \frac{1}{\lambda} u
\]

\[
FV = fV - \frac{1}{\lambda} u(V)V = fV = \lambda U
\]
which is equation (3.28) here equations (3.18),(3.19),(3.20),(3.21),(3.22),(3.23) are used.

**Lemma 3.4.** Let \( M \) be a transversal hypersurface with Lorentzian \((f, g, u, v, \lambda)\) structure of a Lorentzian almost para contact manifold \( M \) endowed with a quarter symmetric non metric connection. Then

\[
(\nabla_X \phi)Y = ((\nabla_X f)Y - u(Y)HX + \lambda fX u(Y) + h(X, Y)U + u(X)\eta(X)U) + ((\nabla_X u)Y + h(X, fY) + u(X)\eta(fX) + 2u(X)u(Y)\lambda)N \\
\nabla_X \xi = \nabla_X V - \lambda HX + \lambda^2 fX + [h(X, V) - u(X) + X(\lambda)]N
\]

(3.32) (3.33)
\[
(\nabla_X \phi) N = -\nabla_X U + fHX - \lambda f^2 X - [h(X, U) - \mu(HX)] N \tag{3.34}
\]
\[
(\nabla_X \eta) Y = (\nabla_X v) Y + h(X, Y) \lambda + u(X) \eta(Y) \lambda \tag{3.35}
\]

for all \(X, Y \in TM\). The proof is straightforward and hence omitted.

**Theorem 3.5.** Let \(M\) be a transversal hypersurfaces with Lorentzian \((f, g, u, v, \lambda)\)–structure of a Lorentzian cosymplectic manifold \(\overline{M}\) endowed with a quarter symmetric non metric connection. Then
\[
(\nabla_X f) Y = u(Y)HX - \lambda f Xu(Y) - [h(X, Y) + u(X)\eta(Y)] U \tag{3.36}
\]
\[
(\nabla_X u) Y = -h(X, fY) - u(X)\eta(fY) - 2u(X)u(Y)\lambda \tag{3.37}
\]
\[
\nabla_X V = \lambda HX - \lambda^2 fX \tag{3.38}
\]
\[
h(X, V) = -X(\lambda) + u(X) \tag{3.39}
\]
\[
\nabla_X U = fHX - \lambda f^2 X \tag{3.40}
\]
\[
h(X, U) = \mu(HX) \tag{3.41}
\]

for all \(X, Y \in TM\).

**Proof.** Using (2.8), (3.12), (3.15) in (3.31), we obtain
\[
((\nabla_X f) Y - u(Y)HX + \lambda f Xu(Y) + h(X, Y)U + u(X)\eta(Y)U)
\]
\[
+((\nabla_X u) Y + h(X, fY) + u(X)\eta(fX) + 2u(X)u(Y)\lambda) N = 0
\]

Equating tangential and normal parts in the above equation, we get (3.36) and (3.37) respectively. Using (2.9) and (3.15) in (3.33), we have
\[
\nabla_X V - \lambda HX + \lambda^2 fX + [h(X, V) + X(\lambda) - u(X)] N = 0
\]

Equating tangential and normal parts we get (3.38) and (3.39) respectively. Using (2.8), (3.14) and (3.15) in (3.34) Using (2.8), (3.14) and (3.15) in (3.33) and equating tangential, we get (3.40). In the last (3.41) follows from (3.35).

**Theorem 3.6.** If \(M\) be a transversal hypersurface with Lorentzian \((f, g, u, v, \lambda)\)-structure of a Lorentzian cosymplectic manifold endowed with a quarter symmetric non metric connection, then the 2- form \(\Phi\) on \(M\) is given by
\[
\Phi(X, Y) \equiv g(X, fY)
\]

is closed.

**Proof.** From (3.36) we get
\[
(\nabla_X \Phi)(Y, Z) = h(X, Y)u(Z) - h(X, Z)u(Y),
\]
\[
(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y) = 0.
\]
Hence the theorem is proved.

**Theorem 3.7.** Let $M$ be a transversal hypersurface with Lorentzian $(f, g, u, v, \lambda)$-structure of a Lorentzian para Sasakian manifold $\overline{M}$ endowed with a quarter symmetric non metric connection. Then

$$(\nabla_X f)Y = g(X, Y)V + \eta(X)\eta(Y)V + u(Y)HX - \lambda fXu(Y)$$

$$-h(X, Y)U - u(X)\eta(Y)U$$

$$(\nabla_X u)Y = g(X, Y)\lambda + \eta(X)\eta(Y)\lambda - h(X, fY) - \eta(fY)u(X) - 2u(X)u(Y)\lambda$$

$$\nabla_X V = -\lambda^2 fX + HX\lambda$$

$$h(X, V) = u(X) - X(\lambda)$$

$$\nabla_X U = -\lambda\eta(X)V + fHX - \lambda f^2X$$

$$h(X, U) = -\lambda^2\eta(X) + \mu(HX)$$

for all $X, Y \in TM$.

**Proof.** Using (2.8), (3.13), (3.15) in (3.32), we obtain

$$g(X, Y)V + \eta(X)\eta(Y)\lambda N + g(X, Y)\lambda N + \eta(X)\eta(Y)\lambda N$$

$$= (\nabla_X f)Y - u(Y)HX + \lambda fXu(Y) + h(X, Y)U + u(X)\eta(Y)U$$

$$+((\nabla_X u)Y + h(X, fY) + \eta(fY)u(X) + 2u(X)u(Y)\lambda)N$$

Equating tangential and normal parts in the above equation, we get (3.42) and (3.43) respectively. Using (2.9) and (3.15) in (3.33), we have

$$\eta(X)\lambda V + \eta(X)\lambda^2 N = \nabla_X V - HX \lambda + \lambda^2 fX$$

$$+[h(X, V) - u(X) + X(\lambda)]N$$

Equating tangential and normal parts we get (3.44) and (3.45) respectively. Using (2.8), (3.14) and (3.15) in (3.33) and equating tangential parts, we get (3.46) in the last (3.47) follows from (3.34).

**Theorem 3.8.** If $M$ is transversal hypersurface with Lorentzian $(f, g, u, v, \lambda)$-structure of a Lorentzian para Sasakian manifold endowed with a quarter symmetric non metric connection, $\overline{M}$. Then $\Phi$ on $M$ given by

$$\Phi(X, Y) = g(X, fY)$$

satisfying

$$(\nabla_X \Phi)(Y, Z) + (\nabla_Y \Phi)(Z, X) + (\nabla_Z \Phi)(X, Y)$$

$$= g(X, Z)v(Y) + g(Y, X)v(Z) + g(Z, Y)v(X) - \lambda u(Z)g(Y, fX)$$

$$-\lambda u(X)g(Z, fY) - \lambda u(Y)g(X, fZ) + \eta(X)v(Y)\eta(Z) + \eta(X)\eta(Y)v(Z)$$
Similarly we obtain,

\[ \nabla_X \phi(X, Z) = g(Y, (\nabla_X f)Z) \]

\[ = g(Y, (g(X, Z) + \eta(X)\eta(Z))V + u(Z)HX - \lambda fXu(Z) \]

\[ - (h(X, Z)U + u(X)\eta(Z))U) \]

\[ = g(X, Z)g(Y, V) + \eta(X)\eta(Z)g(Y, V) + u(Z)g(Y, HX) \]

\[ - \lambda u(Z)g(Y, fX) - h(X, Z)g(Y, U) - u(X)\eta(Z)g(Y, U) \]

\[ = g(X, Z)v(Y) + \eta(X)\eta(Z)v(Y) + u(Z)h(X, Y) \]

\[ - \lambda u(Z)g(Y, fX) - h(X, Z)u(Y) - u(X)\eta(Z)u(Y) \]

We have

\[ (\nabla_X \phi)(Y, Z) = g(Y, (\nabla_X f)Z) \]

\[ = g(Y, (g(X, Z) + \eta(X)\eta(Z))V + u(Z)HX - \lambda fXu(Z) \]

\[ - (h(X, Z)U + u(X)\eta(Z))U) \]

\[ = g(X, Z)g(Y, V) + \eta(X)\eta(Z)g(Y, V) + u(Z)g(Y, HX) \]

\[ - \lambda u(Z)g(Y, fX) - h(X, Z)g(Y, U) - u(X)\eta(Z)g(Y, U) \]

\[ = g(X, Z)v(Y) + \eta(X)\eta(Z)v(Y) + u(Z)h(X, Y) \]

\[ - \lambda u(Z)g(Y, fX) - h(X, Z)u(Y) - u(X)\eta(Z)u(Y) \]

Similarly we obtain,

\[ (\nabla_Y \phi)(Z, X) = g(Y, X)v(Z) + \eta(Y)\eta(X)v(Z) + u(X)h(Y, Z) \]

\[ - \lambda u(X)g(Z, fY) - h(Y, X)u(Z) - u(Y)\eta(X)u(Z) \]

\[ (\nabla_Z \phi)(X, Y) = g(Z, Y)v(X) + \eta(Z)\eta(Y)v(X) + u(Y)h(Z, X) \]

\[ - \lambda u(Y)g(X, fZ) - h(Z, Y)u(X) - u(Z)\eta(Y)u(X) \]

Adding equations (3.48), (3.49) and (3.50), we obtain

\[ (\nabla_X \phi)(Y, Z) + (\nabla_Y \phi)(Z, X) + (\nabla_Z \phi)(X, Y) \]

\[ = g(X, Z)v(Y) + g(Y, X)v(Z) + g(Z, Y)v(X) - \lambda u(Z)g(Y, fX) \]

\[ - \lambda u(X)g(Z, fY) - \lambda u(Y)g(X, fZ) + \eta(X)\eta(Y)v(Z) \]

\[ + \eta(X)\eta(Y)v(Z) + v(X)\eta(Y)\eta(Z) - u(X)u(Y)\eta(Z) - \eta(X)u(Y)u(Z) \]

\[ - u(X)\eta(Y)u(Z) + u(Z)h(X, Y) - u(Y)h(X, Z) + u(X)h(Y, Z) - u(Z)h(Y, X) \]

\[ + u(Y)h(Z, X) - u(X)h(Z, Y) \]

\[ (\nabla_X \phi)(Y, Z) + (\nabla_Y \phi)(Z, X) + (\nabla_Z \phi)(X, Y) \]

\[ = g(X, Z)v(Y) + g(Y, X)v(Z) + g(Z, Y)v(X) - \lambda u(Z)g(Y, fX) \]

\[ - \lambda u(X)g(Z, fY) - \lambda u(Y)g(X, fZ) + \eta(X)\eta(Y)v(Z) \]

\[ + v(X)\eta(Y)\eta(Z) - u(X)u(Y)\eta(Z) - \eta(X)u(Y)u(Z) - u(X)\eta(Y)u(Z). \]
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