T3-Like Finsler Spaces

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Abstract

In 1972, T-tensor in a Finsler space of n-dimensions was introduced and studied simultaneously by H. Kawaguchi [3] and Matsumoto [5]. Several papers related with T-tensor, since then, have been published by various authors namely Hashiguchi [1], Matsumoto [6], Matsumoto and Shimada [7, 8], Rastogi [10, 11] and others. The purpose of the present paper is to study some properties of T-tensor in a Finsler space of three dimensions. Furthermore, we have defined and studied Finsler spaces Fn, whose T-tensor is of special form and called them T3-like Finsler spaces.

1. Introduction

Let \((\hat{f}^i, m^i, n^i)\) be the Moor’s frame of a Finsler space of three dimensions \(F^3\), where \(\hat{f}^i\) is normalized supporting element : \(y^i = L\hat{f}^i\), \(m^i\) is normalized torsion vector given by \(C^i = Cm^i\) and \(n^i\) is a unit vector orthogonal to both \(m^i\) and \(\hat{f}^i\). The metric and angular metric tensors in \(F^3\) are given by [6]:

\[
g_{ij} = l_i l_j + m_i m_j + n_i n_j, \quad h_{ij} = m_i m_j + n_i n_j \tag{1.1}
\]

while the Cartan’s C-tensor is given by Matsumoto [6] as follows:

\[
C_{ijk} = C_{(1)} m_i m_j m_k - C_{(2)} (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) + C_{(3)} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + C_{(2)} n_i n_j n_k. \tag{1.2}
\]

The T-tensor in a Finsler space of n-dimensions is given by

\[
T_{ijkh} = LC_{ijk} l_h + l_i C_{jkh} + l_j C_{ikh} + l_k C_{ijh} + l_h C_{ijk}. \tag{1.3}
\]
where \( l_h \) denotes v-covariant derivative Matsumoto [6].

2. Cartan's C-Tensor in \( F^3 \)

Let \( C_{ijk} m^k = C_{ij} \) and \( C_{ijk} n^k = *C_{ij} \), then from equation (1.2) we can obtain

\[
C_{ij} = C_{(1)} m_i m_j - C_{(2)} (m_i n_j + m_j n_i) + C_{(3)} n_i n_j
\]

(2.1)

and

\[
C_{ij} = C_{(2)} (n_i n_j - m_i m_j) + C_{(3)} (m_i n_j + m_j n_i),
\]

(2.2)

i.e., these tensors are symmetric in lower indices.

Further from equations (2.1) and (2.2), we can obtain

\[
C_{ij} m^j = C_{(1)} m_i - C_{(2)} n_i, \quad *C_{ij} m^j = -C_{(2)} m_i + C_{(3)} n_i,
\]

(2.3)

\[
C_{ij} m^i m^j = C_{(1)}, \quad C_{ij} m^i n^j = -C_{(2)}, \quad C_{ij} n^i n^j = C_{(3)}
\]

(2.4)

and

\[
*C_{ij} m^i n^j = -C_{(2)}, \quad *C_{ij} m^i m^j = C_{(3)}, \quad *C_{ij} n^i n^j = C_{(2)}.
\]

(2.5)

In a three dimensional Finsler space \( F^3 \), v-covariant derivatives of the vectors \( l^i, m^i \) and \( n^i \) are given in Matsumoto [6] as follows:

\[
L l_i^j = h_i^j, \quad L m_i^j = -l_i^j m_j + n^i v_j, \quad L n_i^j = -l_i^j n_j - m^i v_j \]

(2.6)

where \( v_j = v^j_{2\gamma} e_{\gamma ji} \), therefore from equations (2.1) and (2.2) we can respectively obtain

\[
C_{ij} l^i = C_{(1)} l^i m_i m_j - C_{(2)} l^i (m_i n_j + m_j n_i) + C_{(3)} l^i n_i n_j
\]

\[
+ L^{-1}(C_{(1)} l^i (m_i (-l_j m_h + n_j v_h) + m_j (-l_i m_h + n_i v_h))
+ C_{(2)} (m_i (l_j n_h + m_j v_h) + m_j (l_i n_h + m_i v_h)) + n_i (l_j m_h - n_j v_h)
+ n_j (l_i m_h - n_i v_h)) - C_{(3)} (l_i l_j n_h + m_j v_h) + n_j (l_i n_h + m_i v_h))
\]

(2.7)

and

\[
*C_{ij} l^i = C_{(2)} l^i n_i n_j - m_i m_j) + C_{(3)} l^i (m_j n_j + m_j n_i)
\]

\[
+ L^{-1}(C_{(2)} l^i (m_i (-l_j m_h - n_j v_h) + m_j (-l_i m_h - n_i v_h))
- n_i (l_j n_h + m_j v_h) - n_j (l_i n_h + m_i v_h)) - C_{(3)} (l_i l_j n_h + m_j v_h)
\]
+ m_j (l_i n_h + m_j v_h) + n_i (l_j m_h - n_j v_h) + n_j (l_i n_h - n_i v_h))]. \quad (2.8)

Equations (2.7) and (2.8) respectively give

\[ \dot{\mathbf{C}}_{ijh}^i + L^{-1} \mathbf{C}_{i0}^i = 0, \]
\[ \dot{\mathbf{C}}_{ijh}^h = C_{(1)}^0 m_i m_j - C_{(2)}^0 (m_i n_j + m_j n_i) + C_{(3)}^0 n_i n_j, \]
\[ \dot{\mathbf{C}}_{ijh}^m = L^{-1} l_i (C_{(2)}^0 n_h - C_{(1)}^0 m_h) + m_i (C_{(1)}^0 h + 2L^{-1} C_{(2)}^0 v_h) \]
\[ + n_i \{L^{-1} (C_{(1)} - C_{(3)}) v_h - C_{(2)}^0 h \}, \]
\[ \dot{\mathbf{C}}_{ijh}^n = -L^{-1} l_i (C_{(2)}^0 m_h + C_{(3)}^0 n_h) + m_i \{L^{-1} (C_{(1)} - C_{(3)}) v_h - C_{(2)}^0 h \} \]
\[ + n_i \{C_{(3)}^0 h - 2L^{-1} C_{(2)}^0 v_h \}, \]

and

\[ \dot{\mathbf{C}}_{ijh}^m m^h m^i = 2L^{-1} C_{(2)}^0 v_{232} + C_{(1)}^0 h m^h, \]
\[ \dot{\mathbf{C}}_{ijh}^m m^h n^i = L^{-1} (C_{(1)} - C_{(3)}) v_{232} - C_{(2)}^0 h m^h, \]
\[ \dot{\mathbf{C}}_{ijh}^m n^h m^i = 2L^{-1} C_{(2)}^0 v_{233} + C_{(1)}^0 h n^h, \]
\[ \dot{\mathbf{C}}_{ijh}^m n^h n^i = L^{-1} (C_{(1)} - C_{(3)}) v_{2323} - C_{(2)}^0 h n^h, \]
\[ \dot{\mathbf{C}}_{ijh}^m m^h m^i = -2L^{-1} C_{(3)}^0 v_{232} - C_{(2)}^0 h m^h, \]
\[ \dot{\mathbf{C}}_{ijh}^m m^h n^i = -2L^{-1} C_{(2)}^0 v_{232} + C_{(3)}^0 h m^h, \]
\[ \dot{\mathbf{C}}_{ijh}^m n^h m^i = -2L^{-1} C_{(3)}^0 v_{233} - C_{(2)}^0 h n^h, \]
\[ *C_{ij|\ h} \ m^i \ n^i = -2L^{-1} \ C_{(2)} \ v_{233} + C_{(3)}^{|\ h} \ n^h. \quad (2.10) \]

Since \( C_{(1)} + C_{(3)} = C \), therefore from equation (2.10), we can obtain

\[ \{ (C_{ij|\ h} \ m^i + *C_{ij|\ h} \ n^j) \ m^j - C_{|\ h} \} \ m^h = 0, \quad (2.11) \]

\[ \{ (C_{ij|\ h} \ n^i + *C_{ij|\ h} \ m^j) \ m^j \ m^i = L^{-1} \ C_{(1)} \ v_{232} \} \quad (2.12) \]

and

\[ 2 *C_{ij|\ h} \ m^i (C_{(3)} \ n^i - C_{(2)} \ m^i) = (C_{(2)}^2 + C_{(3)}^2)_{|\ h}. \quad (2.13) \]

Hence, we have:

**Theorem (2.1).** In a three dimensional Finsler space \( F^3 \), \( C_{ij|\ h} \) and \( *C_{ij|\ h} \) satisfy equations (2.11), (2.12) and (2.13).

If we take \( h \)-covariant derivative of equations (2.1) and (2.2) and use \( \dot{t}^i = 0 \), \( m^i_j = n^i \ h_j \), \( n^i_j = -m^i \ h_j \), we get

\[ C_{ij|\ k} = (C_{(1)|\ k} + 2C_{(2)} \ h_k) \ m_i \ m_j + (C_{(3)|\ k} - 2C_{(2)} \ h_k) \ n_i \ n_j \]

\[ + \{ (C_{(1)} - C_{(3)}) \ h_k - C_{(2)|\ k} \} \ (m_i \ n_j + m_j \ n_i) \quad (2.14) \]

and

\[ *C_{ij|\ k} = (C_{(3)|\ k} - C_{(2)} \ h_k) \ (n_i \ m_j + m_i \ n_j) + (C_{(2)|\ k} + 2C_{(3)} \ h_k) \ (n_i \ n_j - m_i \ m_j), \quad (2.15) \]

which leads to

\[ (*C_{ij|\ k} \ m^i - C_{ij|\ k} \ n^j) \ n^i - C_{(2)} \ h_k = 0, \quad (2.16) \]

\[ C_{ij|\ k} (m^i \ m^i + n^i \ n^i) = C_{ik}, \quad (2.17) \]

and

\[ C_{ij|\ k} \ m^i \ n^i - *C_{ij|\ k} \ m^i \ m^i = C_{ij|\ k} \ m^i \ n^i + *C_{ij|\ k} \ n^i \ n^i = C \ h_k. \quad (2.18) \]

Hence, we have

**Theorem (2.2).** In a three dimensional Finsler space \( F^3 \), \( h \)-covariant derivative of \( C_{ij} \) and \( *C_{ij} \) satisfy equations (2.16), (2.17) and (2.18).

In case of a \( P^* \)-Finsler space \( F^3 \), Izumi [2], \( P_{ijk} = \lambda A_{ijk} \), therefore we can obtain

\[ C_{ij|l} = \lambda \ C_{ij} + h_0 \ C_{ij}, \quad *C_{ij|l} = \lambda \ *C_{ij} - h_0 \ 'C_{ij}. \quad (2.19) \]
Multiplying equation (2.19) by \( n_j \) and using (2.1) and (2.2), we can obtain on simplification

\[
C_{(1)0} = \lambda C_{(1)} - 3 h_0 C_{(2)}, \quad C_{(2)0} = \lambda C_{(2)} + (C_{(1)} - 2C_{(3)}) h_0, \\
C_{(3)0} = \lambda C_{(3)} + 3 h_0 C_{(2)}.
\]  
(2.20)

Hence, we have

**Theorem (2.3).** In a three dimensional \( P^*-\text{Finsler} \) space \( F^3 \), \( C_{(1)0} \), \( C_{(2)0} \) and \( C_{(3)0} \) are respectively given by (2.20).

Furthermore, from equation (2.19), we can obtain with the help of equations (2.4) and (2.5)

\[
\begin{align*}
'C_{ij0} m^i m^j &= C_{(1)0} + 2C_{(2)} h_0, \\
'C_{ij0} m^i n^j &= -C_{(2)0} + h_0 (C_{(1)} - C_{(3)}), \\
'C_{ij0} n^i n^j &= C_{(3)0} - 2C_{(2)} h_0.
\end{align*}
\]  
(2.21)

Hence, we have

**Theorem (2.4).** In a three dimensional \( P^*-\text{Finsler} \) space \( F^3 \), \( 'C_{ij0} \) satisfies equation (2.21).

Further from equation (2.19), we can obtain on simplification

\[
'C_{ij} = (h_0^2 + \lambda^2)^{-1} (\lambda 'C_{ij0} - h_0 *C_{ij0})
\]  
(2.22)

and

\[
* C_{ij} = (h_0^2 + \lambda^2)^{-1} (\lambda *C_{ij0} + h_0 'C_{ij0})
\]  
(2.23)

which lead to

**Theorem (2.5).** In a \( P^*-\text{Finsler} \) space \( F^3 \), \( 'C_{ij} \) and \( *C_{ij} \) respectively satisfy equations (2.22) and (2.23).

3. **v-Covariant Derivative of C-Tensor in \( F^3 \)**

Taking v-covariant derivative of (1.2) and using equation (2.5) we can obtain on simplification

\[
C_{ijk} \mid_h = C_{(1)} \mid_h m_i m_j m_k - C_{(2)} \mid_h (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j - n_i n_j n_k).
\]
\[ + C_{(3)h} \left( m_i n_j n_k + m_j n_k n_i + m_k n_i n_j \right) \]
\[- L_i^j \bar{\Sigma}_{(i, j, k)} [C_{(1)h} (l_i m_h - n_i v_h) m_j m_k - C_{(2)h} (l_i (m_h m_j n_k + m_j m_k n_h + m_h m_j n_k - n_h m_j n_i - 2m_i n_k n_j + n_i n_j n_k)] \]
\[ + C_{(3)h} \left( l_i (m_h n_j n_k + m_j n_k n_i + m_k n_i n_j) + 2m_i m_j v_h n_k \right) \]
\[ \] (3.1)

which by virtue of symmetry of \( \bar{C}_{ijkh} \) in \( k \) and \( h \) easily leads to
\[ \zeta_{(k, h)} [(C_{(1)h}) m_i m_j m_k + C_{(2)h} (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + C_{(3)h} \left( l_i m_h + n_i v_h \right)] + L_i^j [C_{(1)} (m_i n_j + m_j n_i) m_h v_h + m_i m_j (l_i m_h + n_i v_h)] - C_{(2)} (m_i n_j + m_j n_i) (m_h l_i + 3n_h v_h) \]
\[ + (m_i m_j - n_i n_j) (l_i n_k + 3v_k m_h) + C_{(3)} (m_i n_j + m_j n_i) (n_k l_i + 2v_k m_h) \]
\[ + (m_i m_j - n_i n_j) 2v_k n_h + n_i n_j (l_i n_k + v_h n_k) = 0. \] (3.2)

Multiplying equation (3.2) by \( m_i^j m_i^j m_i^k \), we get on simplification
\[ \alpha_h - \alpha_k m^k m_h + \beta_k m^k n_h = 0, \] (3.3)

where \( \alpha_h \) and \( \beta_h \) are given by
\[ \alpha_h = L C_{(1)h} + C_{(1)} l_i h + 3C_{(2)} v_h \] (3.4)

and
\[ \beta_h = L C_{(2)h} + C_{(2)} l_i h - (C_{(1)} - 2C_{(3)}) v_h. \] (3.5)

With the help of equations (3.3), (3.4) and (3.5), we can obtain
\[ \alpha_0 = \alpha_h \] (3.6)
\[ \beta_0 = \beta_h \] (3.6)
\[ \alpha_h m^h = L C_{(1)h} m^h + 3C_{(2)} v_2^32, \] (3.7)
\[ \alpha_h n^h = L C_{(1)h} n^h + 3C_{(2)} v_2^33, \] (3.8)
\[ \beta_h m^h = L C_{(2)h} m^h - (C_{(1)} - 2C_{(3)}) v_2^32, \] (3.9)
\[ \beta_h n^h = L C_{(2)h} n^h - (C_{(1)} - 2C_{(3)}) v_2^33, \] (3.10)

and
\[ \text{LC}_{(1)} h_n^h + 3 \text{C}_{(2)} v_{2,33} + \text{LC}_{(2)} h_n^h m^h - (\text{C}_{(1)} - 2 \text{C}_{(3)}) v_{2,32} = 0. \]  

(3.11)

Hence, we have

**Theorem (3.1).** In a three dimensional Finsler space \( F^n \), the coefficients \( \text{C}_{(1)} \), \( \text{C}_{(2)} \) and \( \text{C}_{(3)} \) satisfy equations (3.6) to (3.11).

Multiplying equation (3.2) by \( n^i n^j n^k \), we get on simplification

\[ \text{LC}_{(2)} h + \text{C}_{(2)} l_n^h + 3 \text{C}_{(3)} v_h - (\text{LC}_{(3)} n_k^h - 3 \text{C}_{(2)} v_{2,33}) m_h - (\text{LC}_{(2)} h n^k + 3 \text{C}_{(3)} v_{2,33}) n_h = 0. \]  

(3.12)

Multiplying equation (3.12) by \( m^h \), we get

\[ 3(\text{C}_{(2)} v_{2,33} + \text{C}_{(3)} v_{2,32}) = L(\text{C}_{(3)} h n^h - \text{C}_{(2)} h m^h). \]  

(3.13)

Hence, we have

**Theorem (3.2).** In a three dimensional Finsler space \( F^n \), the coefficients \( \text{C}_{(2)} h \) and \( \text{C}_{(3)} h \) satisfy equation (3.13).

Multiplying equation (3.2) by \( n^i n^k m^j m^h \), we get on simplification

\[ 3 \text{C}_{(2)} v_{2,32} + (\text{C}_{(1)} - 2 \text{C}_{(3)}) v_{2,33} = L(\text{C}_{(3)} h m^h + \text{C}_{(2)} h n^h). \]  

(3.14)

Hence, we have

**Theorem (3.3).** In a three dimensional Finsler space \( F^3 \), the coefficients \( \text{C}_{(2)} h \) and \( \text{C}_{(3)} h \) satisfy equation (3.14).

From equations (3.13) and (3.14), on simplification we can obtain

\[ L[\text{C}_{(2)} \text{C}_{(3)} \{ \log (\text{C}_{(3)} \text{C}_{(2)}^{-1}) \} h n^h - (1/2) (\text{C}_{(2)}^2 + \text{C}_{(3)}^2) h m^h] \]

\[ = v_{2,33} (3 \text{C}_{(2)}^2 + 2 \text{C}_{(3)}^2 - \text{C}_{(1)} \text{C}_{(3)}). \]  

(3.15)

Hence, we have

**Theorem (3.4).** In a three dimensional Finsler space \( F^n \), the coefficients \( \text{C}_{(1)} \), \( \text{C}_{(2)} \) and \( \text{C}_{(3)} \) satisfy equation (3.15).

Multiplying equation (3.1) by \( g^{ij}(x, y) \) and using \( \text{C}_{(1)} + \text{C}_{(3)} = \text{C} \), we can obtain on
simplification

\[ C_{k h} = C_{l h} m_k - L^{-1} C(l_h m_l - (1/4) n_k n_l). \]  

(3.16)

Hence, we have

**Theorem (3.5).** In a three dimensional Finsler space F^3, scalar C satisfies (3.16).

4. **P-Tensor in F^3**

In a three dimensional Finsler space F^3, P-tensor is expressed as

\[
P_{ijk} = L[(C_{(1)0} + 3C_{(3)} \ h_0) m_i m_j m_k - C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0] \\
(m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) + (C_{(3)0} - 3C_{(2)} h_0) \\
(m_i n_j n_k + m_j n_k n_i + m_k n_i n_j) + (C_{(2)0} + 3C_{(3)} h_0) n_i n_j n_k. \]  

(4.1)

Let \( P_{ijk} m^k = \cdot P_{ij} \) and \( P_{ijk} n^k = \ast P_{ij} \), then from equation (4.1), we can obtain

\[
\cdot P_{ij} = L[(C_{(1)0} + 3C_{(2)} \ h_0) m_i m_j - C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0] \\
(m_i n_i + m_j n_j) + (C_{(3)0} - 3C_{(2)} h_0) n_i n_j. \]  

(4.2)

and

\[
\ast P_{ij} = L[-(C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0) m_i m_j + (C_{(3)0} - 2C_{(2)} h_0) \\
(m_i n_j + m_j n_i) + (C_{(2)0} + 3C_{(3)} h_0) n_i n_j. \]  

(4.3)

which are symmetric tensors in lower indices. Further from equations (4.2) and (4.3), we can obtain

\[
\cdot P_{ij} m^i = L[(C_{(1)0} + 3C_{(2)} \ h_0) m_i - (C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0) n_i] \]  

(4.4)

\[
\cdot P_{ij} n^i = L[-(C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0) m_i + (C_{(3)0} - 3C_{(2)} h_0) n_i] \]  

(4.5)

\[
\cdot P_{ij} m^i m^i = L[C_{(1)0} + 3C_{(2)} \ h_0], \quad \cdot P_{ij} m^i n^i = -L(C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0) \]  

(4.6)

\[
\cdot P_{ij} n^i n^i = L(C_{(3)0} - 3C_{(2)} h_0), \]  

(4.7)

\[
\ast P_{ij} m^i = L[-(C_{(2)0} - (C_{(1)} - 2C_{(3)}) h_0) m_i + (C_{(3)0} - 3C_{(2)} h_0) n_i], \]  

(4.8)

\[
\ast P_{ij} n^i = L[(C_{(3)0} - 3C_{(2)} h_0) m_i + (C_{(2)0} + 3C_{(3)} h_0) n_i], \]  

(4.9)
\[ \*P^i_j m^j n^i = -L \{ C_{(2)lj0} - (C_{(1)} - 2C_{(3)}) h_0 \}, \]  
(4.10)

\[ \*P^i_j m^j n^i = L(C_{(3)lj0} - 3C_{(2)} h_0), \]  
(4.11)

\[ \*P^i_j n^i n^i = L(C_{(2)lj0} + 3C_{(3)} h_0). \]  
(4.12)

From \( C_{ijk} m^k = \*C_{ij} \) and \( C_{ijk} n^k = \*C_{ij} \), we can easily obtain

\[ \*P^i_j = L(\*C_{ij} l_0 - h_0 \*C_{ij}), \quad \*P^j_i = L(\*C_{ij} l_0 + h_0 \*C_{ij}). \]  
(4.13)

Hence, we have

**Theorem (4.1).** In a three dimensional Finsler space, tensors \( P^i_j \) and \( \*P^i_j \) are related with \( \*C_{ij} \) and \( \*C_{ij} \) by equation (4.13).

In case of a \( P^* \)-Finsler space \( F^3 \), with the help of equation (2.19) and (4.13), we can establish

**Theorem (4.2).** In a three dimensional \( P^* \)-Finsler space \( F^3 \), \( P^i_j = L \lambda \*C_{ij} \) and \( \*P^i_j = L \lambda \*C_{ij} \).

5. **T-Tensor in \( F^3 \)**

Substituting the value of \( C_{ijk} \) and \( C_{ijk} l_0 \) from equations (1.2) and (3.1) in (1.3), we can obtain

\[ T_{ijk} = L \{ C_{(1)i} m_j m_k - C_{(2)i} (m_j m_j n_k + m_j n_k n_i + m_k m_i n_j + m_i n_i n_j) \]  
+ \( C_{(3)i} \) \( m_i n_j n_k + m_j n_k n_i + m_k n_i n_j \}\}

\[ + C_{(1)} [m_j m_k l_h + v_h (m_j n_j n_k + m_j m_k n_i + m_k m_i n_j)] \]  
\[ - C_{(2)} [v_h (m_i n_i n_k + m_j n_k n_i + m_k n_i n_j) - 3 m_i m_j m_k] \]  
\[ + l_h \{ n_i n_j n_k - (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) \} \]  
\[ + C_{(3)} \{ l_h (m_i n_i n_k + m_j n_k n_i + m_k n_i n_j) \} \]  
\[ - 2 v_h (m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) \}. \]  
(5.1)

Equation (5.1) can also be expressed as
\[ T_{ijkh} = m_i m_j m_k \alpha_h - \Sigma_{(i,j,k)} \{ m_i m_j n_k \beta_h - m_i n_j n_k \gamma_h \} + n_i n_j n_k \delta_h \] (5.2)

where

\[ \gamma_h = LC_{(3)h} + C_{(3)} I_h + C_{(2)} v_{h^*} \]
\[ \delta_h = LC_{(2)h} + C_{(2)} I_h \] (5.3)

such that

\[ \gamma_0 = \gamma_h^h = LC_{(3)0} + C_{(3)^*} \]
\[ \delta_0 = \delta_h^h = \beta_0 = LC_{(2)0} + C_{(2)} \]
\[ \gamma_h m^h = LC_{(3)h} m^h + C_{(2)} v_{232} \]
\[ \delta_h m^h = LC_{(2)h} m^h \]
\[ \gamma_h n^h = LC_{(3)h} n^h + C_{(2)} v_{233} \]
\[ \delta_h n^h = LC_{(2)h} n^h \] (5.4)

Multiplying equation (5.2) by \( g_{jk}^h \), we can obtain \( LC_{l_0} = -C \), \( LC_{(1)l_0} = -C_{(1)} \) and \( LC_{(2)l_0} = -C_{(2)} \). Hence, we have

**Theorem (5.1).** In a three dimensional Finsler space \( F^3 \), if T-tensor is expressed by (5.2), coefficients \( \alpha_0, \beta_0, \gamma_0, \delta_0, C_{(1)}, C_{(2)} \) and \( C_{(3)} \) satisfy \( \alpha_0 + \gamma_0 = 0, \beta_0 = \delta_0, \)

\( LC_{l_0} = -C, LC_{(1)l_0} = -C_{(1)} l_0 = -C_{(1)} \) and \( LC_{(2)l_0} = -C_{(2)} \).

From equation (5.2), we can obtain by virtue of \( T_{ijkh} g^{kh} = T_{ij} \)

\[ T_{ij} = \{ LC_{(1)h} m^h + 3C_{(2)} v_{232} - LC_{(2)h} n^h + (C_{(1)} - 2C_{(3)}) v_{233} \} m_i m_j \]
\[ + \{ LC_{(3)h} n^h + C_{(2)} v_{233} - LC_{(2)h} m^h + (C_{(1)} - 2C_{(3)}) v_{232} \} \]
\[ (m_i n_j + m_j n_i) + \{ LC_{(3)h} m^h + C_{(2)} v_{232} + LC_{(2)h} n^h \} n_i n_j \] (5.5)

which satisfies \( T_{ij} \delta^i = 0, \)

\[ T_{ij} m^i = \{ LC_{(1)h} m^h + 3C_{(2)} v_{232} - LC_{(2)h} n^h + (C_{(1)} - 2C_{(3)}) v_{233} \} m_j \]
\[ + \{ LC_{(3)h} n^h + C_{(2)} v_{233} - LC_{(2)h} m^h + (C_{(1)} - 2C_{(3)}) v_{232} \} \]
and

\[ T_{ij} n^i = \{ LC_{(3)h} m^h + C_{(2)} v_{232} + LC_{(2)h} n^h \} n_j \]
\[ + \{ LC_{(3)h} n^h + C_{(2)} v_{233} - LC_{(2)h} m^h + (C_{(1)} - 2C_{(3)}) v_{232} \} m_j \] (5.6)

From equation (5.5), we can further obtain by virtue of \( T_{ij} g^{ij} = T \)

\[ T = LC_{l} m^l + 4C_{(2)} v_{232} + (C_{(1)} - 2C_{(3)}) v_{233} \] (5.8)
6. **T3-Like Finsler Spaces**

From equation (5.2), we can write tensor $T_{ijkh}$ of a 3-dimensional Finsler space $F^3$ in the following form

$$T_{ijkh} = \sum_{(i,j,k)} (a_{hk} h_{ij} + b_{hk} m_i m_j),$$

(6.1)

where $a_{hk}$ and $b_{hk}$ are the second order tensors defined by

$$a_{hk} = \gamma_h m_k + (1/3) \delta_h n_k, \quad b_{hk} = ((1/3) \alpha_h - \gamma_h) m_k - (\beta_h + (1/3) \delta_h) n_k.$$  

(6.2)

From equation (6.2), we can obtain

$$a_{0k} = \gamma_0 m_k + (1/3) \delta_0 n_k, \quad b_{0k} = ((1/3) \alpha_0 - \gamma_0) m_k - (\beta_0 + (1/3) \delta_0) n_k$$

(6.3)

and

$$a_{0k} + b_{0k} = (1/3) \alpha_0 m_k - \beta_0 n_k.$$  

(6.4)

Comparing equations (1.3) and (6.1) and solving, we get

$$4 a_{0k} = -b_{0l} (\delta^i_k + 2m^i m_k).$$  

(6.5)

Hence, we have

**Theorem (6.1).** In a 3-dimensional Finsler space $F^3$, second order tensors $a_{0k}$ and $b_{0k}$ satisfy (6.4) and (6.5).

In any three dimensional Finsler space $F^3$ the T-tensor is defined by (6.1), which helps us give the following definition:

**Definition (6.1).** A Finsler space $F^n$ ($n > 3$), shall be called T3-like Finsler space, if for arbitrary second order tensors $a_{hk}$ and $b_{hk}$ satisfying $a_{h0} = 0, b_{h0} = 0$, its T-tensor $T_{ijkh}$ is non-zero and is expressed by an equation of the form:

$$T_{ijkh} = \sum_{(i,j,k)} (a_{hk} h_{ij} + b_{hk} C_i C_j).$$

(6.6)

It is known, Shimada [13], that for second curvature tensor $P_{hijk}$, the Ricci tensors defined by

$$P_{hk}^{(1)} = P^i_{hijk} = C_{klh} - C^j_{hklj} + P^j_{kr} C^r_{jrh} - P^r_{hk} C_r$$

(6.7)

and

$$P_{hk}^{(2)} = P^i_{hjk} = C_{klh} - C^j_{hklj} + C^r_{kh} C_{rl0} - P^j_{hr} C^r_{kj}$$

(6.8)
are non-symmetric such that
\[ P_{h0}^{(1)} = 0, \quad P_{h0}^{(2)} = 0, \quad P_{0k}^{(1)} = C_{k0}^{(1)} = P_k, \quad P_{0k}^{(2)} = C_{k0} = P_k. \] (6.9)

If we assume that the tensor \( a_{hk} = P_{hk}^{(1)} \), we can obtain by virtue of equation (6.7) and \( *T_{kh} = T_{ijkh} g^{ij} \)
\[ *T_{kh} = (n + 1) P_{hk}^{(1)} + C^2 b_{hk} + 2 b_{hi} C^i C_k. \] (6.10)

Since \( *T_{kh} \) is symmetric in \( k \) and \( h \), from equation (6.10), we can obtain
\[ \zeta_{(k, h)} \{ C_{kh} + P_{ihr} C^r_{jh} - (n + 1)^{-1} b_{ki} (C^2 \delta^i_h + 2 C^i C_h) \} = 0, \] (6.11)
therefore, we can have

**Theorem (6.2).** In a T3-like Finsler space \( F_n \) \((n > 3)\), if tensor \( a_{hk} = P_{hk}^{(1)} \), equation (6.11) is satisfied.

Multiplying equation (6.11) by \( L^k \), we get
\[ L(b_{0h} C^2 + 2 b_{0i} C^i C_h) + (n + 1) P_h = 0, \] (6.12)
and
\[ b_{0i} C^i = -(n + 1)/3} L^{-1} C^{-2} P_i C^i. \] (6.13)

From equations (6.12) and (6.13), we can obtain
\[ b_{0h} = (n + 1) P_i L^{-1} C^{-2} \{(2/3) C^i C_h - \delta^i_h\}. \] (6.14)

Hence, we have

**Theorem (6.3).** In a T3-like Finsler space \( F^n \) \((n > 3)\), if the tensor \( a_{hk} = P_{hk}^{(1)} \), the tensor \( b_{0h} \) is given by equation (6.14).

From equation (6.10), we can obtain on simplification
\[ b_{hi} C^i = [L C_{rh} C^i + C^2 l_h - (n + 1) P_{hi}^{(1)} C^i]/3C^2. \] (6.15)
Substituting the value of \( b_{hi} C^i \), from equation (6.15) in (6.10), we obtain on simplification
\[ b_{hk} = C^{-2} \{ [LC_{rh} - (n + 1) P_{hi}^{(1)} (\delta^i_k - (2/3C^2) C^i C_k) + l_k C_h + (1/3) l_h C_k] \}. \] (6.16)
Hence, we have

**Theorem (6.4).** In a T3-like Finsler space $F^n$ (n > 3), if the tensor $a_{hk}$ is given by the Ricci tensor $P_{hk}$, the tensor $b_{hk}$ is given by equation (6.16).

### 7. C-Reducible Finsler Space

In a C-reducible Finsler space $F^3$, it is known that Matsumoto [4], $C_{(1)} = (3/4) C$, $C_{(2)} = 0$ and $C_{(3)} = (1/4) C$, therefore from equations (2.1) and (2.2), we can obtain

$$'C_{ij} = (C/4) (3 m_i m_j + n_i n_j)$$

and

$$^*C_{ij} = (C/4) (m_i n_j + m_j n_i),$$

which lead to

$$'C_{ij} m^j = (3/4) C m_i, \quad 'C_{ij} n^j = (1/4) C n_i = {^*C_{ij} m^j, \quad ^*C_{ij} n^j = (1/4) C m_i.}$$

Furthermore, we can obtain

$$'C_{ij}^h = (1/4) C_{ij} (3 m_i m_j + n_i n_j) + L^{-1}(1/4) C [3(m_i l_j m_h + n_j v_h)$$

$$+ m_j (-l_i m_h + n_i v_h)] - (n_i (l_i n_h + m_j v_h) + n_j (l_i n_h + m_i v_h))$$

and

$$^*C_{ij}^h = (1/4) [C_{ij} (m_i n_j + m_j n_i) - L^{-1} C (m_i l_j n_h + m_j v_h)$$

$$+ m_i (l_i n_h + m_i v_h) + n_i (l_j m_h - n_j v_h) + n_j (l_i n_h - n_i v_h)].$$

From equations (3.1) and (3.2), we can obtain

$$C_{ijk}^h = (1/4) C_{ij} (3 m_i m_j m_k + (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)$$

$$- L^{-1} \Sigma_{i,j,k} (1/4) C (l_i m_j - n_i v_h) m_j m_k$$

$$+ \{l_i (m_h n_j n_k + m_k n_h n_j + m_k n_h n_k) + 2 m_i m_j v_h n_k],$$

which by virtue of symmetry of $C_{ijk}^h$ in $k$ and $h$, easily leads to

$$\zeta_{(k, h)} \{[C_{(1)}^h m_i m_j m_k + C_{(3)}^h (m_i n_j n_k + m_j n_k n_i + m_k n_i n_j)]$$

$$+ L^{-1}[C_{(1)} ((m_i n_j + m_j n_i) m_k v_h + m_i m_j (l_i m_k + n_k v_h)]$$

$$+ C_{(3)} ((m_i n_j + m_j n_i) (n_k l_h + 2 v_k m_h) + (m_i m_j - n_i n_j) 2 v_k n_h)$$

involves.
\[ + n_i n_j (l_{h, m_k} + v_{h, n_k}) \] = 0. \quad (7.7)

Equations (3.4) to (3.11) for a C-reducible Finsler space reduce to

\[
\begin{align*}
\alpha_h &= (3/4) (Lc_{h} + C_{h}), \quad \beta_h = (1/4) C v_h, \\
\alpha_0 &= \alpha_h^h = (3/4) (Lc_{0} + C), \quad \beta_0 = \beta_h^h = 0, \quad \alpha_h m^h = (3/4) Lc_{h} m^h \\
\alpha_h n^h &= (3/4) Lc_{h} n^h, \quad \beta_h m^h = (1/4) C v_{232}, \quad \beta_h n^h = (1/4) C v_{233} \\
3Lc_{h} n^h + C v_{232} &= 0. \quad (7.8)
\end{align*}
\]

For a C-reducible Finsler space equations (3.12), (3.13) and (3.14) can be expressed as

\[ 3C(v_h - v_{233} n_h) = Lc_k n^k m_h, \quad (7.9) \]
\[ Lc_{h} n^h - 3C v_{232} = 0 \quad (7.10) \]
and
\[ Lc_{h} m^h = C v_{233}. \quad (7.11) \]

From equations (7.8) and (7.10), we can obtain

\[ Cl_h n^h = 0 \quad \text{or} \quad v_{232} = 0. \quad (7.12) \]

Hence, we have

**Theorem (7.1).** In a 3-dimensional C-reducible Finsler space \( Lc_{h} m^h = C v_{233} \) and \( Cl_h n^h = 0. \)

From equation (4.2) for a C-reducible Finsler space \( F^3 \), we get

\[ T_{ijkh} = (1/4) \Sigma_{(i,j,k)} [(Lc_{h} + C_{h})(m_i m_j m_k + m_i n_j n_k) + C v_{h} m_i m_j n_k], \quad (7.13) \]
which leads to

\[ T_{ij} = C v_{233} (m_i m_j + (1/4) n_i n_j), \quad (7.14) \]
\[ T_{ij} m_i = C v_{233} m_j, \quad T_{ij} n^i = (1/4) C v_{233} n_j \quad (7.15) \]
and
\[ T = (5/4) C v_{233}. \quad (7.16) \]

Hence, we have
Theorem (7.2). In a 3-dimensional C-reducible Finsler space Tijkh is given by (7.13) and \( T = (5/4) C v_{ij0}^{33} \).

8. P-Reducible Finsler Spaces

A Finsler space is defined as P-reducible by Matsumoto and Shimada [7] and studied by Rastogi and Kawaguchi [9] and Rastogi [12], if its torsion tensor \( P_{ijk} = A_{ijk}^l = 0 \), is expressible as

\[
P_{ijk} = (n + 1)^{-1} (A_{kl0} h_{ij} + A_{li0} h_{jk} + A_{jl0} h_{ki}).
\]

(8.1)

In three dimensional Finsler space \( P_{ijk} \) on simplification can be expressed as

\[
P_{ijk} = L [ (C_{(1)lj0} + 3C_{(2)} h_0 ) (m_i m_j m_k + (1/4) C_{li0} (n_i n_j n_k + n_j n_k n_i + n_k n_i n_j) + m_k n_i n_j ) + (1/4) C_{ij0} (3n_i n_j n_k + m_i m_j n_k + m_j m_k n_i + m_k m_i n_j) ).
\]

(8.2)

Multiplying (8.2) by \( g^{jk} \) and using \( P_i = LC_{il0} \) and \( C_{i0} = C_{i0}^l m^l \), we get

\[
3 C_{i0} = 4C_{(1)lj0} + 12 C_{(2)} h_0 \quad \text{or} \quad 4C_{(3)lj0} - C_{i0} = 12 C_{(2)} h_0
\]

(8.3)

Hence, we have

Theorem (8.1). In a P-reducible Finsler space \( F^3 \), coefficients \( C_{(1)}, C_{(2)} \) and \( C_{(3)} \) are related by equation (8.3).

From equation (2.1), we can easily obtain

\[
L 'C_{ij0} = (1/4) [(3m_i m_j + n_i n_j) (C_{(1)lj0} + 3C_{(2)} h_0 + (1/4) C_{(i0)}]'
\]

\[
+ C_{ij0} (m_j n_i + m_i n_j). \]

(8.4)

From equation (8.4), we can obtain

\[
' L 'C_{ij0} = 0, \quad L 'C_{ij0} m^j = (1/4) (3 C_{i0} m_j + C h_0 n_i),
\]

\[
L 'C_{ij0} n^j = (1/4) (C_{i0} n_j + C h_0 m_j).
\]

(8.5)

Similarly from equation (2.2), we can obtain

\[
L "C_{ij0} = C_{i0} (m_i n_j + m_j n_i) + 4C h_0 m_i m_j - L h_0 'C_{ij},
\]

(8.6)
which implies

\[ *C_{ij0}^j = 0, \quad L *C_{ij0}^j m^j = m_i (4C - C_{(1)} h_0 + n_i (C_{l0} - C_{(2)} h_0), \]

\[ L *C C_{ij0}^j n^j = (C_{l0} + L h_0 C_{(2)} ) m_i - L h_0 C_{(3)} n_i. \]  

(8.7)

Hence, we have

**Theorem (8.2).** In a P-reducible Finsler space \( F^3 \), tensors \( C_{ij} \) and \( *C_{ij} \) satisfy equations (8.6) and (8.7).

From equation (1.3) with the help of equations (8.1), we can obtain on simplification

\[ L T_{ijk0} = L^2 C_{ij0}^k h_{j0} + (n + 1)^{-1} \{ A_{ii0} \Sigma_{(i, k, h)} (t_i h_{kh}) + A_{j0} \Sigma_{(j, k, h)} (t_j h_{kh}) \]

\[ + A_{k0} \Sigma_{(i, j, k)} (t_i h_{jk}) + A_{j0} \Sigma_{(i, j, k)} (t_j h_{jk}) \}, \]  

(8.8)

which gives

**Theorem (8.3).** In a P-reducible Finsler space \( F^3 \), T-tensor satisfies equation (8.8).

In a T3-like Finsler space equation (8.8) gives on simplification

\[ L \Sigma_{(i, j, k)} \{ a_{n0} h_{ij} + b_{n0} C_i C_j \} - L^2 C_{ijk} h_{j0} - (n + 1)^{-1} \{ A_{ii0} \Sigma_{(i, k, h)} (t_i h_{kh}) \]

\[ - (n + 1)(b_{hk} C_j + b_{hj} C_k) \} + A_{j0} \Sigma_{(i, k, h)} (t_i h_{kh}) - (n + 1)(b_{hk} C_i + b_{hi} C_k) \}

\[ + A_{k0} \Sigma_{(i, j, k)} (t_i h_{jk}) - (n + 1)(b_{hi} C_j + b_{hj} C_i) \} + A_{j0} \Sigma_{(i, j, k)} (t_j h_{jk}) = 0. \]  

(8.9)

Hence, we have

**Theorem (8.4).** If a T3-like Finsler space is also a P-reducible Finsler space it satisfies (8.9).

From equation (8.9), we can obtain

\[ 3 a_{hk0} + b_{h0} (C^2 \delta^i_k + 2C^i C_k) - L C_{k0} h_{j0} - L C_k C_{h0} - L C_{k0} \]

\[ = -2(n + 1)^{-1} C_{i0} (b_{hk} C^i + b^i_j C_k - b_{hj} C^j \delta^i_k). \]  

(8.10)

Hence, we have
Theorem (8.5). In a T3-like P-reducible Finsler space arbitrary tensors $a_{hk}$ and $b_{hk}$ are related by equation (8.10).

References


