In the present paper, we have introduced the concept of \((\alpha, \beta, \gamma)\) – metric and find some important tensors for \((\alpha, \beta, \gamma)\) – metric, where 
\[
\alpha = a_{ij}(x) y^i y^j / 2, \quad 1\text{-form} \quad \beta = b_i(x) y^i, \quad \text{cubic metric} \quad \gamma = a_{ijk}(x) y^i y^j y^k / 3.
\]
We have also considered the hypersurface given the equation \(b(x) = \text{constant}\) of the Finsler space with the \((\alpha, \beta, \gamma)\) – metric given by \(L = L(\alpha, \beta, \gamma)\).

Keywords: Finsler Space with \((\alpha, \beta, \gamma)\) – metric; cubic metric; one form metric; angular metric tensor; fundamental tensor and reciprocal tensor.

1. Introduction

Matsumoto, M. in the year 1972 \(^5\), introduced the notion of \((\alpha, \beta)\) – metric and studied in detail. A Finsler metric \(L(x, y)\) is called an \((\alpha, \beta)\) – metric, if it is positively homogenous function of degree one in Riemannian metric \(\alpha = \{a_{ij}(x) y^i y^j\}^{1/2}\) and 1-form \(\beta = b_i(x) y^i\). The well-known examples of \((\alpha, \beta)\) – metric are Rander’s metric \(\alpha + \beta\) \(^{10}\), Kropina metric \(\alpha^2 / 2\) \(^3\), generalized Kropina metric \(\frac{\alpha^{m+1}}{2m} (m \neq 0, -1)^\frac{1}{3}\), and Matsumoto metric \(\frac{\alpha}{\alpha - \beta}\) \(^7\) etc, whose studies have greatly contributed to the growth of Finsler geometry.

Again in the year 1979, Matsumoto, M. \(^4\) introduced the concept of cubic metric on a differentiable manifold with the local co-ordinates, defined by 

\[
L(x, y) = \{a_{ijk}(x) y^i y^j y^k\}^{1/3}.
\]

where, \(a_{ijk}(x)\) are components of a symmetric tensor field of \((0, 3)\) -type depending on the position \(x\) alone, and a Finsler space with a cubic metric is called the cubic Finsler space.
After that several authors also studied the cubic Finsler spaces\textsuperscript{3, 5, 12, 13, 14, 15}. In the year 2011, Pandey, T. N. and Chaubey, V. K.,\textsuperscript{9} had introduced the concept of \((\gamma, \beta)\) – metric and a number of propositions and theorems obtained, where \(\gamma = \{a_{ijk}(x) y^i y^j y^k\}^{1/3}\) is a cubic metric and \(\beta = b_i(x) y^i\) is a one-form metric.

After studying these valuable research papers, we have introduced the \((\alpha, \beta, \gamma)\)– metric, where \(\alpha = \{a_{ij}(x) y^i y^j\}^{1/2}\), 1-form \(\beta = b_i(x) y^i\) and cubic-metric, \(\gamma = \{a_{ijk}(x) y^i y^j y^k\}^{1/3}\). In the year 1995, Matsumoto, M.,\textsuperscript{6} had discussed the properties of special hypersurface of Rander space with \(b_i(x) = (\partial i b)\) being the gradient of a scalar function \(b_i(x)\) and also consider a hypersurface which is given by \(b_i(x) = \text{constant}\).

In this paper we have also considered the hypersurface given by the equation \(b_i(x) = \text{constant}\), of the Finsler space with \((\alpha, \beta, \gamma)\)– metric.

\section{Basic tensors of \((\alpha, \beta, \gamma)\) – metric}

\textbf{Definition :} A Finsler metric \(L(x, y)\) is called a \((\alpha, \beta, \gamma)\) – metric, when \(L\) is positively homogenous function \(L(\alpha, \beta, \gamma)\) of first degree in the variables \(\alpha, \beta\) and \(\gamma\), where \(\gamma = \{a_{ijk}(x) y^i y^j y^k\}^{1/3}\) is a cubic metric and \(\beta = b_i(x) y^i\) is a one-form metric.

In this present paper we have used the following results
\[
a_{ijk}(x) y^i y^k = a_i, \quad a_{ijk} y^k = a_{ij} a_{ij} b_j = b_i, \quad a_i a_i = \gamma^3
\]
where, \((a^{ij})\) is the inverse matrix of \((a_{ij})\).

As for \((\alpha, \beta, \gamma)\) – metric,
\[
L = L(\alpha, \beta, \gamma) \quad \text{(2.1)}
\]

Where,
\[
\alpha = \{a_{ij}(x) y^i y^j\}^{1/2}, \quad \beta = b_i(x) y^i \quad \text{and} \quad \gamma = \{a_{ijk}(x) y^i y^j y^k\}^{1/3} \quad \text{(2.2)}
\]

Differentiating (2.2), we get,
\[
\frac{\partial \alpha}{\partial y^r} = \frac{\partial \alpha}{\partial y}, \quad \text{where} \quad a_{ir} y^r = y_i, \quad b_r = \frac{\partial \beta}{\partial y^r} \quad \text{and} \quad \frac{\partial \gamma}{\partial y^r} = \frac{\partial \gamma}{\partial y^r} \quad \text{(2.3)}
\]

Again differentiating (2.1) with respect to \(y^r\), we get,
\[
l_i = \partial_i L, \quad \text{where} \quad \partial_i L = \frac{\partial L}{\partial y^i}
\]
\[
l_i = \frac{L_{\alpha}}{\alpha} y_i + L_{\beta} b_i + \frac{L_{\gamma}}{\gamma^2} a_i \quad \text{(2.4)}
\]
Further subscripts \( \alpha, \beta, \gamma \) denote partial differentiations with respect to \( \alpha, \beta, \gamma \) respectively.

Again differentiating (2.4) with respect to \( y^j \), the angular metric tensor 
\[ h_{ij} = L \partial_i \partial_j L \] is given by

\[
h_{ij} = P^*_0 a_{ij}(x) + P_{-1} a_{ij}(x, y) + q^*_2 y_i y_j + q^*_{-1} (y_i b_j + y_j b_i) + q^*_{-3} (a_i y_j + a_j y_i)
+ q_{-2} (a_i b_j + a_j b_i) + q_{-4} a_i a_j + q_0 b_i b_j \]  

(2.5)

Where,

\[
P^*_0 = \frac{L a_0}{\alpha}, \quad P_{-1} = \frac{2L a_0}{\gamma}, \quad q^*_{-2} = \frac{L}{\gamma} (L_{\alpha\alpha} - \frac{L a_0}{\alpha}),
q^*_{-1} = \frac{L a_0}{\alpha}, \quad q^*_3 = \frac{L a_0}{\alpha\gamma}, \quad q_{-2} = \frac{L a_0}{\gamma}, \quad q_{-4} = \frac{L}{\gamma} \left( L_{\gamma\gamma} - \frac{2L a_0}{\gamma} \right),
q_0 = L L_{\beta\beta},
\]

In (2.5) the subscripts of coefficients \( P^*_0, P_{-1}, q^*_{-2}, q^*_{-1}, q^*_3, q_{-2}, q_{-4} \) and \( q_0 \) are used to indicate respective degrees of homogeneity.

Again ,

\[
g_{ij} = h_{ij} + l_i l_j
\]

\[
g_{ij} = a_{ij}(x) P^*_0 + P_{-1} a_{ij}(x, y) + \left( q^*_{-2} + \frac{L^2}{\alpha^2} \right) y_i y_j
+ \left( q^*_{-1} + \frac{L a_0 L a_0}{\alpha} \right) (y_i b_j + y_j b_i) + \left( q^*_{-3} + \frac{L a_0 L a_0}{\alpha\gamma} \right) (a_i y_j + a_j y_i)
+ \left( q_{-2} + \frac{L a_0 L a_0}{\gamma} \right) (a_i b_j + a_j b_i) + \left( q_{-4} + \frac{L^2}{\gamma^2} \right) a_i a_j
+ (q_0 + L^2 b_i b_j)
\]

If ,

\[
(q^*_{-2} + \frac{L^2}{\alpha^2}) = P^*_{-2}, \quad (q^*_{-1} + \frac{L a_0 L a_0}{\alpha}) = P^*_{-1},
(q^*_{-3} + \frac{L a_0 L a_0}{\alpha\gamma}) = P^*_{-3}, \quad (q_{-2} + \frac{L a_0 L a_0}{\gamma}) = P_{-2},
\]
\[
\left( q + \frac{L^2}{\gamma^2} \right) = P_{-4}, \quad \left( q_0 + L_0^2 \right) = P_0,
\]
then, we have,
\[
g_{ij} = P_0^* a_{ij}(x) + P_{-1} a_{ij}(x, y) + P_{-2} y_i y_j + P_{-4} (y_i b_j + y_j b_i) \\
+ P_{-3} (a_i y_j + a_j y_i) + P_{-2} (a_i b_j + a_j b_i) + P_{-4} a_i a_j + P_0 b_i b_j
\]
Since we know that \( \frac{\partial \gamma}{\partial y_i} = a_i \gamma \) and from \( \frac{\partial \gamma}{\partial \eta} = \frac{y_i}{\gamma} \), then we get \( a_i = \gamma y_i \). By using \( a_i = \gamma y_i \), we find,
\[
g_{ij} = P_0^* a_{ij}(x) + P_{-1} a_{ij}(x, y) + a_i a_j \left( P_{-2} \gamma^{-2} + 2 \frac{P_{-4}}{\gamma^2} \right) \\
+ \left( \frac{P_{-1}}{\gamma} + P_{-2} \right) (a_i b_j + a_j b_i) + P_0 b_i b_j
\]
where we put,
\[
S_{-4} = P_{-2} \gamma^{-2} + 2 \frac{P_{-4}}{\gamma^2} + P_{-4}
\]
\[
S_{-2} = \frac{P_{-2}}{\gamma} + P_{-2}
\]
then we have,
\[
g_{ij} = P_0^* a_{ij}(x) + P_{-1} a_{ij}(x, y) + S_{-4} a_i a_j + S_{-2} (a_i b_j + a_j b_i) + P_0 b_i b_j \tag{2.6}
\]
We know that,
\[
g^{hj} g_{ij} = \delta^h_i
\]
Then ,the reciprocal tensor of \( g_{ij} \) is given by,
\[
g^{ij} = \frac{a_i a_j}{\gamma} - \frac{a^i a^j}{\gamma} \left( \frac{\pi_{-3} S_{-4} - \pi_{-2} S_{-2}}{\gamma} \right) - b^i b^j \left( \frac{\pi P_0 - \pi S_{-2}}{\gamma} \right)
\]
\[-a'^{i}b'^{j}\frac{(\pi_{-1}S_{-2} - \tau_{-1}P_{0})}{J} - a^{i}b^{j}\frac{(\tau S_{-2} - \pi S_{-4})}{J}\]

Where, \( J = P_{0}^{*} + P_{-1}, \quad \tau = P_{0}^{*} + P_{-1} + \gamma^{3}S_{-4} + S_{-2}\beta \)

\[\pi = S_{-2}\gamma^{3} + P_{0}\beta, \quad \tau_{-1} = \beta S_{-4} + S_{-2}b^{2}\]

\[\pi_{-1} = P_{0}^{*} + P_{-1} + S_{-2}\beta + P_{0}b^{2}, \quad d = \frac{1}{\pi_{-1} - \pi_{-1}}\]

\[g^{ij} = S_{1}a'^{i}a'^{j} - S_{2}a'^{i}a'^{j} - S_{3}b'^{i}b'^{j} - S_{4}(a'^{i}b'^{j} + a'^{j}b'^{i})\]

(2.7)

Where,

\[S_{1} = \frac{1}{J}, \quad S_{2} = \frac{(\pi_{-1}S_{-4} - \tau_{-1}S_{-2})}{J},\]

\[S_{3} = \frac{(\tau P_{0} - \pi S_{-4})}{J}, \quad S_{4} = \frac{(\pi_{-1}S_{-2} - \tau_{-1}P_{0})}{J} = \frac{(\tau S_{-2} - \pi S_{-4})}{J}\]

where,

\[(\pi_{-1}S_{-2} - \tau_{-1}P_{0}) = (\tau S_{-2} - \pi S_{-4})\]

\[(\pi_{-1}S_{-2} - \tau_{-1}P_{0}) = (\tau S_{-2} - \pi S_{-4}) = \frac{P_{0}^{*}P_{-1}}{\gamma} + \frac{P_{-1}P_{0}^{*}}{\gamma} + \beta\left(\frac{P_{0}^{*}}{\gamma}\right)^{2} +

2\beta\frac{P_{0}^{*}P_{-1}}{\gamma} + P_{0}^{*}P_{-2} + P_{-1}P_{-2} + \beta(P_{-2})^{2} - \beta\frac{P_{0}^{*}P_{-1}}{\gamma} - 2\beta\frac{P_{0}^{*}P_{-1}}{\gamma} - \beta P_{0}P_{-4}\]

**Theorem (2.1)** The angular metric tensor \( h_{ij} \), the fundamental tensor \( g_{ij} \) and its reciprocal tensor \( g^{ij} \) of \((\alpha, \beta, \gamma)\) – metric are given by equations (2.5), (2.6) and (2.7) respectively.

### 3. The Hypersurfaces \( F^{n-1}(c) \)
In this section we have considered a special \((\alpha, \beta, \gamma)\) – metric with a gradient \(b_i(x) = \partial_i b\) for a scalar function \(b(x)\) and consider a hypersurface \(F^{n-1}(c)\) which is given by the equation \(b(x) = c\) (constant).

Since the parametric equation of \(F^{n-1}(c)\) is \(x^i = x^i(u^\alpha)\), hence, \((\partial/\partial u^\alpha) b(x(u)) = b_i(x) X^i_\alpha\), where \(b_i(x)\) are considered as covariant components of a normal vector field of \(F^{n-1}(c)\). Therefore, along the \(F^{n-1}(c)\), we have,

\[ b_i X^i_\alpha = 0 \quad \text{and} \quad b_i y^i = 0 \quad (3.1) \]

In general, the induced metric \(L(u, v)\) given by,

\[ L(u, v) = L \left( \left( a_{\alpha\beta} (u) v^\alpha v^\beta \right)^{\frac{1}{2}}, \left( a_{\alpha\beta\gamma} (u) v^\alpha v^\beta v^\gamma \right)^{\frac{1}{3}} \right), \]

\[ (3.2) \]

where,

\[ a_{\alpha\beta} (u) = a_{ij} (x(u)) X^i_\alpha X^j_\beta \quad \text{and} \quad a_{\alpha\beta\gamma} (u) = a_{ijk} (x(u)) X^i_\alpha X^j_\beta X^k_\gamma \]

By using equation (3.1) and (2.7), we have,

\[ g^{ij} b_i b_j = b^2 (S_1 - S_3 b^2), \quad \text{where} \quad b^2 = a^{ij} b_i b_j \]

Hence we get,

\[ b_i = b \sqrt{S_1 - b^2 S_3} N_i \]

\[ (3.3) \]

Hence from (2.7) and (3.3) we get,

\[ b^i = a^{ij} b_j = \frac{b}{\sqrt{S_1 - b^2 S_3}} N^i + \left( \frac{b^2 S_1}{S_1 - b^2 S_3} \right) a^i \]

\[ (3.4) \]
Theorem (3.1). Let $F^n$ be a Finsler space with $\alpha, \beta, \gamma$ – metric (2.1) and $b_i(x) = \partial b(x)$ and $F^{n-1}(c)$ be a hypersurface of $F^n$ given by $b(x) = c$ (constant). If the Riemannian metric $a_{ij}(x) \, dx^i dx^j$ be positive definite and $b_i$ is a non-zero field, then the induced metric of $F^{n-1}(c)$ is a Riemannian metric given by (3.2) and relations (3.3) and (3.4) hold.

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