Metallic structure on Lagrangian manifold

Geeta Verma

Department of Mathematics
Shri Ramswaroop memorial group of professional colleges,
Lucknow, India.

Email: geeta_verma153@rediffmail.com

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In this paper, the author convention with the Lagrange vertical structure on the vertical space $T_V(E)$ endowed with a non null $(1,1)$ tensor field $F$ satisfying metallic structure $F^2 - \alpha F - \beta I = 0$. The horizontal subspace $T_H(E)$ is applied on the same structure. Next, some theorems are proved and obtained conditions under which the distribution $L$ and $M$ are $\nabla$-parallel, $\bar{\nabla}$ anti half parallel when $\nabla = \bar{\nabla}$. Lastly, certain theorems on geodesics on the Lagrange manifold are deduced.

Keywords: Metallic structure; Lagrangian manifold; vertical space.

1. Introduction

Let $M$ and $E$ be two differentiable manifolds of dimension $n$ and $2n$ respectively and $(E, \pi, M)$ be vector bundles with $\pi(E) = M$. The local coordinate systems $(x^1, x^2, \ldots, x^n)$ about $x$ in $M$ and $(y^1, y^2, \ldots, y^n)$ about $y$ in $E$. The induced coordinates in $\pi^{-1}(U)$ are $(x^i, y^\alpha)$, $1 \leq i \leq n, 1 \leq \alpha \leq n$ where $U$ is a coordinate neighborhood in $M$. The canonical basis for tangent space $T_u(E)$ at $u \in \pi^{-1}(U)$ is \( \left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha} \right\} \) or simply $\{ \partial_i, \partial_\alpha \}$ where $\partial_i = \frac{\partial}{\partial x^i}$ etc. If $(x^h, x^\alpha)$ be coordinates of a point in the interesting region $\pi^{-1}(U) \cap \pi^{-1}(U)$, we can write

\begin{align*}
x^{i'} &= x^i (x^i) \\
y^{\alpha'} &= \frac{\partial x^{\alpha'}}{\partial x^i} y^\alpha
\end{align*}

and another canonical basis in the intersecting region are given by

\begin{align*}
\partial_{i1} &= \frac{\partial x^i}{\partial x^{i'}} \partial_i \\
\partial_{\alpha1} &= \frac{\partial y^\alpha}{\partial y^{\alpha'}} \partial_\alpha
\end{align*}
The tangent space of \( E \) is denoted by \( T(E) \) and spanned by \( \{ \partial_i, \partial_\alpha \} \) and its subspaces by \( T_V(E) \) and \( T_H(E) \) spanned by \( \{ \partial_\alpha \} \) and \( \{ \partial_i \} \) respectively. Obviously

\[
\dim T_V(E) = \dim T_H(E) = n
\]

Let us suppose that the Riemannian material structure on \( T(E) \) is given by

\[
G = g_{ij}(x^i, y^\alpha)dx^i \otimes dx^j + g_{ab}(x^i, y^\alpha)\delta y^a \otimes \delta y^b
\]

where \( g_{ij}(x^i, y^\alpha) = g_{ij}(x^i) \), \( g_{ab} = \frac{1}{2}\partial_a \partial_b(x^i, y^\alpha) \) and \( L(x^i, y^\alpha) \) the Lagrange function. We call such a manifold as Lagrangian manifold.

If \( X \in T(E) \), we can write

\[
X = \bar{X}^i \partial_i + X^\alpha \partial_\alpha
\]

The automorphism \( P : \chi(T(E)) \to \chi(T(E)) \) defined by

\[
PX = \bar{X}^i \partial_i + X^\alpha \partial_\alpha
\]

is a natural almost product structure on \( T(E) \) i.e. \( P^2 = I \), \( I \) unit tensor field. If \( v \) and \( h \) are the projection morphisms of \( T(E) \) onto \( T_V(E) \) and \( T_H(E) \) respectively, then

\[
P_0 h = v_0 P
\]

2. Metallic Structure

Let \( T_V(E) \) be the vertical space and there exists a non-null tensor field \( F_v \) of type (1,1) satisfying

\[
F_v^2 - \alpha F_v - \beta I = 0
\]

where \( \alpha, \beta \) are positive integers, we say that \( T_V(E) \) admits metallic structure. In this case \( \text{rank} \ (F_v) = r \) which is constant every where. Let us call \( F_v \) as Lagrange vertical structure on \( T_V(E) \).

Theorem 1. If Lagrange vertical structure \( F_v \) is defined on the vertical space \( T_V(E) \), it is possible to define similar structure on the horizontal subspace \( T_H(E) \) with the help of the almost product structure of \( T(E) \).

Proof: Let us put

\[
F_h = P F_v P
\]

then \( F_h \) is a tensor field of type (1,1) on \( T_H(E) \). Also

\[
F_h^2 = (P F_v P)(P F_v P) = P F_v^2 P
\]

as \( P \) is an almost product structure on \( T(E) \).

Similarly \( F_h^3 = P F_v^3 P \) and so on. Thus, we have by virtue of (9)
Thus, $F_h$ gives metallic structure on $T_H(E)$.

**Theorem 2.** If Lagrange vertical structure $F_v$ of rank $r$ be defined on $T_V(E)$, the similar type of structure can be defined on the enveloping space $T(E)$ with the help of projection morphism of $T(E)$.

**Proof:** Since Lagrange structure $F_v$ is defined on $T_V(E)$, the Lagrange horizontal structure $F_h$ is induced on $T_H(E)$ by theorem (2.1). If $v$ and $h$ are projection morphisms of $T_V(E)$ and $T_H(E)$ on $T(E)$, let us put

$$F = Fvh + Fv$$  \hspace{1cm} (12)

As $hv = vh = 0$ and $h^2 = h, v^2 = v$, we have

$$F^2 = F^2h + F^2v$$

Thus

$$F^2 - \alpha F - \beta I = (F^2h - \alpha Fh - \beta I)h$$
$$+ (F^2v - \alpha Fv - \beta I)v$$
$$= 0$$  \hspace{1cm} (13)

Making use of equations (9) and (11).

Hence

$$F^2 - \alpha F - \beta I = 0$$

Since $rank(F_v) = rank(F_h) = r$, hence $rank(F) = 2r$.

Let us define tensor fields $l$ and $m$ of type (1,1) on $T(E)$ with metallic structure of rank $2r$ as follows

$$l = \frac{(F^2 - \alpha F)}{\beta}$$
$$m = I - \frac{(F^2 - \alpha F)}{\beta}$$  \hspace{1cm} (14)

Then it is easy to show that

$$l + m = I$$
$$l^2 = l, m^2 = m, lm = ml = 0,$$  \hspace{1cm} (15)
$$Fl = lF = F, Fm = mF = 0.$$  \hspace{1cm} (16)

This implies that the operators $l$ and $m$ when applied to the tangent space are complementary projection operators $^{3,7,7}$. 
3. Parallelism of distributions

Let $E$ be a 2n-dimensional Lagrangian manifold with metallic structure on $T(E)$, then there exist complementary distributions $L$ and $M$ corresponding to complementary projection operators $l$ and $m$. Let $\nabla$ and $\bar{\nabla}$ be defined as follows

$$\bar{\nabla}_X Y = l\nabla_X (lY) + m\nabla_X (mY)$$  \hspace{1cm} (17)$$

and

$$\nabla_X Y = m\nabla_X (mY) + l\nabla_X (lY) + [lX, mY]$$ \hspace{1cm} (18)$$

It can be shown easily that $\bar{\nabla}$ and $\nabla$ are linear connections on $E$.

**Definition 3.1** The distribution $L$ is called $\nabla$-parallel if for all $X \in L, Y \in T(E)$ the vector field $\nabla_X Y \in L$.

**Definition 3.2** The distribution $L$ will be said $\nabla$-half parallel if for all $X \in L, Y \in T(E)$, $(\Delta F)(X, Y) \in L$ where

$$(\Delta F)(X, Y) = F\nabla_X Y - F\nabla_Y X - \nabla_{FX} Y + \nabla_Y (FX)$$ \hspace{1cm} (19)$$

**Definition 3.3** The distribution $L$ is called $\nabla$-anti half parallel if for all $X \in L, Y \in T(E)$, $(\Delta F)(X, Y) \in M$.

Now we prove the following theorems.

**Theorem 3.** On the metallic structure manifold, the distribution $L$ and $M$ are $\bar{\nabla}$ as well as $\nabla$-parallel.

**Proof:** Since $lm = ml = 0$, hence from (17) and (18), we have

$$m\nabla_X Y = m\nabla_X (mY)$$

If $Y \in L, mY = 0$ so $m\nabla_X Y = 0$ Therefore $\bar{\nabla}_X Y \in L$. Hence for $Y \in L, X \in T(E)$ $\Rightarrow \bar{\nabla}_X Y \in L$. So $L$ is $\bar{\nabla}$-parallel.

Similarly for $X \in T(E), Y \in L

$$\bar{\nabla}_X Y = m\nabla_X (mY) + m[lX, mY] = 0$$ as $mY = 0$.

So $\nabla_X Y \in L$. Hence $L$ is $\nabla$-parallel.

In a similar manner, $\bar{\nabla}$ and $\nabla$ parallelism of $M$ can also be proved.

**Theorem 4.** On the metallic structure manifold, the distribution $L$ and $M$ are $\nabla$-parallel if and only if $\bar{\nabla}$ and $\nabla$ are equal.

**Proof:** If $L$ and $M$ are $\nabla$-parallel then $\forall X, Y \in T(E)$,

$$m\nabla_X (lY) = 0, l\nabla_X (mY) = 0.$$ 

Therefore, since $l + m = I$,

$$\nabla_X (lY) = l\nabla_X (lY)$$

and

$$\nabla_X (mY) = m\nabla_X (mY)$$
So,

\[ \nabla_X Y = l \nabla_X (lY) + m \nabla_X (mY) = \nabla_X Y \]

Hence \( \nabla = \bar{\nabla} \)
The converse of the theorem proved easily.

**Theorem 5.** On the metallic structure manifold, \( E \), the distribution \( M \) is \( \bar{\nabla} \)-anti half parallel if for all \( X \in M, Y \in T(E) \)

\[ m \bar{\nabla}_Y (FX) = m \nabla_{FX} mY. \]

**Proof:** Since \( Fm = mF = 0 \), using the equation (19) for connection \( \bar{\nabla} \), we have

\[ m(\Delta F)(X, Y) = m \bar{\nabla}_Y FX - m \nabla_{FX} Y \tag{20} \]

In view of the equation (17), we have

\[ \bar{\nabla}_{FX} Y = l \nabla_{FX} (lY) + m \nabla_{FX} (mY) \]

\[ m \bar{\nabla}_{FX} Y = m \nabla_{FX} (mY) \text{ as } lm = 0, m^2 = m \]

\[ m(\Delta F)(X, Y) = m \bar{\nabla}_Y FX - m \nabla_{FX} Y \]

As \( (\Delta F)(X, Y) \in L \) so \( m(\Delta F)(X, Y) = 0 \). Thus

\[ m \bar{\nabla}_Y (FX) = m \nabla_{FX} (mY), \]

which proves the theorem.

4. Geodesics on the Lagrangian manifold

Let \( \gamma \) be a curve in \( E \) with tangent \( T \). Then \( \gamma \) is called geodesic with respect to connection \( \bar{\nabla} \) if \( \nabla_T T = 0 \).

**Theorem 6.** A curve \( \gamma \) will be geodesic with respect to connection \( \bar{\nabla} \) if the vector fields \( \nabla_T T - \nabla_T (mT) \in M \) and \( \nabla_T (mT) \in L \).

**Proof:** Since \( \gamma \) will be geodesic with respect to connection \( \bar{\nabla} \), hence \( \nabla_T T = 0 \). On making use of the equation (17), the above equation assumes the following form

\[ l \nabla_T (lT) + m \nabla_T (mT) = 0. \]

Since \( l + m = I \) we can write the above equation as

\[ l \nabla_T (I - mT) + m \nabla_T (mT) = 0 \]

or

\[ l \nabla_T T - l \nabla_T (mT) + m \nabla_T (mT) = 0 \]

Therefore \( l(\nabla_T T - \nabla_T (mT)) \) and \( m \nabla_T (mT) = 0 \).
Hence $\nabla_T T - \nabla_T (mT) \in M$ and $\nabla_T (mT) \in L$, which proves the theorem.

**Theorem 7.** The $(1,1)$ tensor field $l$ and $m$ are always covariantly constants with respect to connection $\nabla$.

**Proof:** \(\forall X, Y \in T(E)\), we have

\[
(\nabla_X l)(Y) = \nabla_X (lY) - l\nabla_X Y.
\]  

(21)

Making use of equation (17), we get

\[
(\nabla_X l)(Y) = l\nabla_X (l^2 Y) + m\nabla_X (mlY) - l\{l\nabla_X lY + m\nabla_X mY\}
\]

Since $l^2 = l$, $m^2 = m$, $lm = ml = 0$, we get

\[
(\nabla_X l)(Y) = l\nabla_X (lY) - l\nabla_X lY = 0.
\]

So, $l$ is covariantly constant. The fact that $m$ is covariantly constant can be proved analogously.

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