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Some Properties of CR-Submanifolds of a Kenmotsu Manifold

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The purpose of the present paper is to initiate the study of CR-submanifolds of a Kenmotsu manifold and obtain some properties. The conditions under which the distributions required by CR- submanifolds to be integrable, are obtained. D-parallel normal section of CR-submanifolds have been studied.

Keywords: CR-submanifold; Kenmotsu manifold; ξ-vertical and mixed totally geodesic.

1. Introduction

In 1978, Bejancu introduced the notion of CR-Submanifold of Kaehler manifold [1]. On the other hand CR-submanifold of a Sasakian manifold have been stidied by Kobayashi [5], Shahid et al.[8], Yano and Kon[11], and others. Bejancu and Papaghuic [2] studied CR-submanifolds of a Kenmotsu manifold. In this paper we obtain certain results in generalised form of [5],[6] and [7].

2. Preliminaries

Let $\overline{M}^{(2n+1)}(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold, where ϕ is (1,1) tensor field, η is a 1-form and g is the Riemannian metric [3],[10]

$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \tag{1}$$

$$\phi^2(X) = -X + \eta(\xi), \tag{2}$$

$$g(X, \xi) = \eta(X), \tag{3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{4}$$

for any vector fields X,Y on $\overline{M}$.

If Moreover

$$(\overline{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad X, Y \in \chi(\overline{M}) \tag{5}$$

$$(\overline{\nabla}_X \xi) = X - \eta(X)\xi \tag{6}$$

Where $\overline{\nabla}$ denotes the Riemannian connection of g, then $(\overline{M}, \phi, \xi, \eta, g)$ is called an almost Kenmotsu manifold [4].

Definition 2.1[9]. An m-dimensional Riemannian submanifold M of a Kenmotsu manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists a differentiable distribution D:$x \in M \rightarrow D_x \subset T_xM$ such that

1. the distribution D_x is invariant under ϕ, that is, $\phi D_x \subset D_x$ for each $x \in M$;

2. the complementary orthogonal distributions $D^\perp$:$x \in M \rightarrow D_x^\perp \subset T_xM$ of D is anti-invariant under ϕ, that is, $\phi D_x^\perp \subset T_x^\perp M$ for all $x \in M$, where T_xM and $T_x^\perp M$ are the tangent space and the normal space of M at x, respectively.

If dim$D_x^\perp$=0(resp., dimD_x=0), then the CR-submanifold is called an invariant (resp., anti invariant) submanifold.The distribution D (resp.,$D^\perp$) is called the horizontal (resp.,vertical) distribution. Also, the pair (D,$D^\perp$) is called ξ-horizontal (resp.,vertical) if $\xi_x \in D_x$(resp.,$\xi_x \in D_x^\perp$) [5]. For any vector field X tangent to M, we put

$$X = PX + QX, \tag{7}$$

Where PX and QX belong to the distributions D and $D^\perp$. For any vector field N normal to M, we put

$$\phi N = BN + CN, \tag{8}$$

Where BN(resp.,CN) denotes the tangential(resp.,normal) component of ϕN. Let $\overline{\nabla}$(resp.,∇ be the covariant differentiation with respect to the Levi-civita connection on $\overline{M}$(resp.,M). The Gauss and Weingarten formulas for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \qquad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N \tag{9}$$

for any X,$Y \in TM$ and $N \in T^\perp M$,where h(resp., A) is the second fundamental form(resp.,tensor)of M in $\overline{M}$, and $\nabla^\perp$ denotes the normal connection. Moreover,we have

$$g(h(X, Y), N) = g(A_N X, Y) \tag{10}$$

3. Some basic lemmas.

First we prove the following lemma.

Lemma 3.1 Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$. Then

$$P(\nabla_X \phi PY) + P(\nabla_Y \phi PX) - P(A_{\phi QY} X) - P(A_{\phi QX} Y)$$

$$= \phi P\nabla_X Y + \phi P\nabla_Y X - \eta(Y)\phi PX + \eta(X)\phi PY, \quad (11)$$

$$Q(\nabla_X \phi PY) + Q(\nabla_Y \phi PX) - Q(A_{\phi QY}X) - Q(A_{\phi QX}Y)$$

$$= 2Bh(X,Y) - \eta(Y)\phi QX + \eta(X)\phi QY, \quad (12)$$

$$h(X,\phi PY) + h(Y,\phi PX) + \nabla^{\perp}_X \phi QY + \nabla^{\perp}_Y \phi QX$$

$$= \phi Q\nabla_X Y + \phi Q\nabla_Y X + 2Ch(X,Y) \quad (13)$$

for any X,$Y \in TM$.

Proof. From the definition of a Kenmotu manifold and using equations (2.7),(2.8), and (2.9),we get

$$\nabla_X \phi PY + h(X,\phi PY) - A_{\phi QY}X + \nabla^{\perp}_X \phi QY - \phi(\nabla_X Y + h(X,Y))$$

$$+\nabla_Y \phi PX + h(Y,\phi PX) - A_{\phi QX}Y + \nabla^{\perp}_Y \phi QX - \phi(\nabla_Y X + h(X,Y))$$

$$= -\phi[\eta(Y)X + \eta(X)Y] \quad (14)$$

for any X,$Y \in TM$. Now using equation (2.7) and equating horizontal,vertical and normal component in equation (3.4), we get the result.

Lemma 3.2 Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$.Then

$$2(\overline{\nabla}_X \phi)(Y) = \nabla_X \phi Y - \nabla_Y \phi X + h(X,\phi Y) - h(Y,\phi X)$$

$$-\phi[X,Y] - \phi[\eta(Y)X + \eta(X)Y] \quad (15)$$

for any X,$Y \in D$.

$$2(\overline{\nabla}_Y \phi)(X) = -\nabla_X \phi Y + \nabla_Y \phi X - h(X,\phi Y) + h(Y,\phi X)$$

$$+\phi[X,Y] - \phi[\eta(Y)X + \eta(X)Y] \quad (16)$$

for any X,$Y \in D$.

Proof. By Gauss formula we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X - h(Y,\phi X) \quad (17)$$

Also, we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) + \phi[X,Y] \quad (18)$$

From equations (3.6) and (3.7),we get

$$(\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) = \nabla_X \phi Y + h(X,\phi Y) - \nabla_Y \phi X$$

$$-h(Y,\phi X) - \phi[X,Y] \quad (19)$$

Also for Kenmotsu manifold,we have

$$(\overline{\nabla}_X \phi)(Y) + (\overline{\nabla}_Y \phi)(X) = -\phi[\eta(Y)X + \eta(X)Y] \quad (20)$$

Adding and Subtracting equations (3.8) and (3.9),the lemma follows.
Lemma 3.4 Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$.Then

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y$$

$$-\phi[Y,Z] - \phi[\eta(Z)Y + \eta(Y)Z] \tag{21}$$

$$2(\overline{\nabla}_Z\phi)(Y) = -A_{\phi Y}Z + A_{\phi Z}Y - \nabla^{\perp}_Y\phi Z + \nabla^{\perp}_Z\phi Y$$

$$+\phi[Y,Z] - \phi[\eta(Z)Y + \eta(Y)Z] \tag{22}$$

for any Y,$Z \in D^{\perp}$.

Proof. From Weingarten formula, we have

$$\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = -A_{\phi Y}Z + \nabla^{\perp}_Z\phi Y + A_{\phi Z}Y - \nabla^{\perp}_Y\phi Z \tag{23}$$

Also,we have

$$\overline{\nabla}_Z\phi Y - \overline{\nabla}_Y\phi Z = (\overline{\nabla}_Z\phi)(Y) - (\overline{\nabla}_Y\phi)(Z) - \phi[Y,Z] \tag{24}$$

From equations (3.13) and (3.14),we get

$$(\overline{\nabla}_Y\phi)(Z) - (\overline{\nabla}_Z\phi)(Y) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y - \phi[Y,Z] \tag{25}$$

Also for Kenmotsu manifold,we have

$$(\overline{\nabla}_Y\phi)(Z) - (\overline{\nabla}_Z\phi)(Y) = -\phi[\eta(Z)Y + \eta(Y)Z] \tag{26}$$

Adding equations (3.15) and (3.16),we get

$$2(\overline{\nabla}_Y\phi)(Z) = A_{\phi Y}Z - A_{\phi Z}Y + \nabla^{\perp}_Y\phi Z - \nabla^{\perp}_Z\phi Y$$

$$-\phi[Y,Z] - \phi[\eta(Z)Y + \eta(Y)Z] \tag{27}$$

Subtracting equations (3.15) and (3.16),we get

$$2(\overline{\nabla}_Z\phi)(Y) = -A_{\phi Y}Z + A_{\phi Z}Y - \nabla^{\perp}_Y\phi Z + \nabla^{\perp}_Z\phi Y$$

$$+\phi[Y,Z] - \phi[\eta(Z)Y + \eta(Y)Z] \tag{28}$$

This proves our assertions.

Lemma 3.4 Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$.Then

$$2(\overline{\nabla}_X\phi)(Y) = -A_{\phi Y}X + \nabla^{\perp}_X\phi Y - \nabla_Y\phi X - h(Y,\phi X)$$

$$-\phi[X,Y] - \phi[\eta(Y)X + \eta(X)Y] \tag{29}$$

$$2(\overline{\nabla}_Y\phi)(X) = A_{\phi Y}X - \nabla^{\perp}_X\phi Y + \nabla_Y\phi X + h(Y,\phi X)$$

$$+\phi[X,Y] - \phi[\eta(Y)X + \eta(X)Y] \tag{30}$$

for any Y,$Z \in D^{\perp}$.

Proof. By using Gauss and Weingarten equations for $X \in D$ and $Y \in D^{\perp}$ respectively we get

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X) \tag{31}$$

Also we have

$$\overline{\nabla}_X \phi Y - \overline{\nabla}_Y \phi X = (\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) + \phi[X, Y] \tag{32}$$

From equations (3.21) and (3.22), we get

$$(\overline{\nabla}_X \phi)(Y) - (\overline{\nabla}_Y \phi)(X) = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y$$

$$-\nabla_Y \phi X - h(Y, \phi X) - \phi[X, Y] \tag{33}$$

Also for Kenmotsu manifold,we have

$$(\overline{\nabla}_Y \phi)(Z) - (\overline{\nabla}_Z \phi)(Y) = -\phi[\eta(Z)Y + \eta(Y)Z] \tag{34}$$

Adding equations (3.23) and (3.24),we get

$$2(\overline{\nabla}_X \phi)(Y) = -A_{\phi Y} X + \nabla^{\perp}_X \phi Y - \nabla_Y \phi X - h(Y, \phi X)$$

$$-\phi[X, Y] - \phi[\eta(Y)X + \eta(X)Y] \tag{35}$$

Subtacting equations (3.23) and (3.24),we get

$$2(\overline{\nabla}_Y \phi)(X) = A_{\phi Y} X - \nabla^{\perp}_X \phi Y + \nabla_Y \phi X + h(Y, \phi X)$$

$$+\phi[X, Y] - \phi[\eta(Y)X + \eta(X)Y] \tag{36}$$

Hence the Lemma.

4. Parallel distributions

Definition 4.1. The horizontal (resp.,vertical) distribution D (resp.,$D^{\perp}$) is said to be parallel [1] with respect to the connection ∇ on M if $\nabla_X Y \in D$(resp.,$\nabla_Z W \in D^{\perp}$) for any vector field X,$Y \in D$ (resp.,W,$Z \in D^{\perp}$).

Now we prove the following proposition.

Propostion 4.2. Let M be a ξ-vertical CR-subamnifold of a Kenmotsu manifold $\overline{M}$. If the horizontal distribution D is parallel, then

$$h(X, \phi Y) = h(Y, \phi X) \tag{37}$$

for all X,$Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

$$\nabla_X \phi Y \in D, \quad \nabla_Y \phi X D \qquad for\ any\ \ X, Y \in D \tag{38}$$

Thus using the fact QX=QY=0 for Y∈D, equation (3.2) gives

$$Bh(X,Y) = g(X,Y)Q\xi \quad for\ any\ \ X,Y \in D \tag{39}$$

Also, since

$$\phi h(X,Y) = Bh(x,y) + Ch(X,Y), \tag{40}$$

then

$$\phi h(X,Y) = g(X,Y)Q\xi + Ch(X,Y)\xi \quad for\ any\ \ X,Y \in D. \tag{41}$$

Next from equation (3.3), we have

$$h(X,\phi Y) + h(Y,\phi X) = 2Ch(X,Y) = 2\phi h(X,Y) - 2g(X,Y)Q\xi, \tag{42}$$

for any X,Y∈D.Putting X=ϕX∈D in equation (4.6),we get

$$h(\phi X,\phi Y) + h(Y,\phi^2 X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi \tag{43}$$

or

$$h(\phi X,\phi Y) - h(Y,X) = 2\phi h(\phi X,Y) - 2g(\phi X,Y)Q\xi. \tag{44}$$

Similary, putting Y=ϕY∈D in equation (4.6),we get

$$h(\phi Y,\phi X) - h(Y,X) = 2\phi h(X,\phi Y) - 2g(X,\phi Y)Q\xi. \tag{45}$$

Hence from equations (4.8) and (4.9), we have

$$\phi h(X,\phi Y) - \phi h(Y,\phi X) = g(X,\phi Y)Q\xi - g(\phi X,Y)Q\xi. \tag{46}$$

Operating ϕ on both sides of equation (4.10) and using $\phi\xi$=0,we get

$$h(X,\phi Y) = h(Y,\phi X) \tag{47}$$

for all X,Y∈D.
Now, for the distribution $D^\perp$,we prove the following proposition.

Propostion 4.3. Let M be a ξ-vertical CR-submanifold of a Kenmotsu manifold $\overline{M}$.If the distribution $D^\perp$ is parallel with respect to the connection on M,then

$$(A_{\phi Y}Z + A_{\phi Z}Y) \in D^\perp \quad for\ any\, Y\ , Z \in D^\perp \tag{48}$$

Proof. Let Y,Z∈$D^\perp$, then using Gauss and Weingarten formula ,we obtain

$$-A_{\phi Z}Y + \nabla^\perp_Y \phi Z - A_{\phi Y}Z + \nabla^\perp_Z \phi Y = \phi\nabla_Y Z + \phi\nabla_Z Y + 2\phi h(Y,Z)$$

$$= -\phi[\eta(Z)Y + \eta(Y)Z] \tag{49}$$

for any YZ∈$D^\perp$. Taking inner product with X∈D in equation (4.13),we get

$$g(A_{\phi Y}Z,X) + g(A_{\phi Z}Y,X) = g(\nabla_Y Z,\phi X) + g(\nabla_Z Y,\phi X). \tag{50}$$

If the distribution $D^{\perp}$ is parallel, then $\nabla_Y Z \in D^{\perp}$ and $\nabla_Z Y \in D^{\perp}$ for any $Y, Z \in D^{\perp}$. So from equation (4.14) we get

$$g(A_{\phi Y}Z, X) + g(A_{\phi Z}Y, X) = 0 \quad or \quad g(A_{\phi Y}Z + A_{\phi Z}Y, X) = 0 \tag{51}$$

which is equivalent to

$$(A_{\phi Z}Z + A_{\phi Z}Y) \in D^{\perp} \quad for\, any\, Y, Z \in D^{\perp}, \tag{52}$$

this completes the proof.

Definition 4.4[5]. A CR-submanifold is said to be mixed totally geodesic if $h(X,Z)$=0 for all $X \in D$ and $Z \in D^{\perp}$.
The following lemma is an easy consequence of (2.7).
Lemma 4.5. Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$. Then M is mixed totally geodesic if and only if $A_N X \in D$ for all $X \in D$.
Definition 4.6[5]. A normal vector field $N \neq 0$ is called D-parallel normal section if $\nabla^{\perp}_X N$=0 for all $X \in D$.

Now we have the following proposition.
Proposition 4.7. Let M be a mixed totally geodesic ξ-vertical CR-subamanifold of a Kenmotsu manifold $\overline{M}$. Then the normal section $N \in \phi D^{\perp}$ is D-parallel if and only if $\nabla_X \phi N \in D$ for all $X \in D$.
Proof. Let $N \in \phi D^{\perp}$. Then from equation (3.2) we have

$$Q(\nabla_Y \phi X) = 0 \quad for\, any\, X \in D,\ Y \in D^{\perp}. \tag{53}$$

In particular,we have $Q(\nabla_Y X)$=0.By using it in equation (3.3),we get

$$\nabla^{\perp}_X \phi QY = \phi Q \nabla_X Y \quad or \quad \nabla^{\perp}_X N = -\phi Q(\nabla_X \phi N) \tag{54}$$

Thus, if the normal section $N \neq 0$ is D-parallel, then using Definition (4.6) and (4.18), we get

$$\phi Q(\nabla_X \phi N) = 0 \tag{55}$$

which is equivalent to $\nabla_X \phi N \in D$ for all $X \in D$.The converse part easily follows from equation (4.18).

5. Integrability conditions of distributions.

First we calculate the Nijenhuis tensor $N_{\phi}(X, Y)$ on a Kenmotsu manifold $\overline{M}$. For this, first we prove the following lemma.
Lemma 5.1. Let $\overline{M}$ be a Kenmotsu manifold,then

$$(\overline{\nabla}_{\phi X}\phi)(Y) = \eta(Y)X - \eta(Y)\eta(X)\xi - \eta(X)\overline{\nabla}_Y \xi$$

$$+\phi(\overline{\nabla}_Y \phi)(X) + \eta(\overline{\nabla}_Y X)\xi \tag{56}$$

for any X,$Y$$\in$$T\overline{M}$.
Proof.From the definition of Kenmotsu manifold $\overline{M}$, we have

$$(\overline{\nabla}_{\phi X}\phi)(Y) = -\eta(Y)\phi^2 X - \eta(\phi X)\phi Y - (\overline{\nabla}_Y \phi)(\phi X) \quad (57)$$

Also,we have

$$\begin{aligned}(\overline{\nabla}_Y \phi)(\phi X) &= \overline{\nabla}_Y \phi^2 X - \phi\overline{\nabla}_Y \phi X \\ &= \overline{\nabla}_Y \phi^2 X - \phi\overline{\nabla}_Y \phi X + \phi(\phi\overline{\nabla}_Y X) - \phi(\phi\overline{\nabla}_Y X) \\ &= -\overline{\nabla}_Y X + \eta(X)\overline{\nabla}_Y \xi - \phi(\overline{\nabla}_Y \phi X - \phi\overline{\nabla}_Y X) - \phi(\phi\overline{\nabla}_Y X)\end{aligned}$$

$$= -\overline{\nabla}_Y X + \eta(X)\overline{\nabla}_Y \xi - \phi(\overline{\nabla}_Y \phi)(X) + \overline{\nabla}_Y X - \eta(\overline{\nabla}_Y X)\xi. \quad (58)$$

Using equation (5.3) in (5.2),we get

$$(\overline{\nabla}_{\phi X}\phi)(Y) = \eta(Y)X - \eta(Y)\eta(X)\xi - \eta(X)\overline{\nabla}_Y \xi + \phi(\overline{\nabla}_Y \phi)(X) + \eta(\overline{\nabla}_Y X)\xi \quad (59)$$

for any X,$Y$$\in$$T\overline{M}$,which completes the proof of the lemma.

On a Kenmotsu manifold $\overline{M}$, Nijenhuis tensor is given by

$$N_\phi(X,Y) = (\overline{\nabla}_{\phi X}\phi)(Y) - (\overline{\nabla}_{\phi Y}\phi)(X) - \phi(\overline{\nabla}_X \phi)(Y) + \phi(\overline{\nabla}_Y \phi)(X) \quad (60)$$

for any X,$Y$$\in$$T\overline{M}$. From equations (5.1) and (5.5),we get

$$\begin{aligned}N_\phi(X,Y) &= \eta(Y)X - \eta(X)Y - \eta(X)\overline{\nabla}_Y \xi + \eta(Y)\overline{\nabla}_X \xi + \eta(\overline{\nabla}_Y X)\xi \\ &\quad -\eta(\overline{\nabla}_X Y)\xi - 2\phi(\overline{\nabla}_X \phi)(Y) + 2\phi(\overline{\nabla}_Y \phi)(X) \quad (61)\end{aligned}$$

Thus using equation (2.3) in above equation and after some calculations,we obtain

$$\begin{aligned}N_\phi(X,Y) &= -\eta(Y)X - 3\eta(X)Y - \eta(X)\overline{\nabla}_Y \xi + \eta(Y)\overline{\nabla}_X \xi + \eta(\overline{\nabla}_Y X)\xi \\ &\quad -\eta(\overline{\nabla}_X Y)\xi + 4\phi(\overline{\nabla}_Y \phi)(X) + 4\eta(Y)\eta(X)\xi. \quad (62)\end{aligned}$$

for any X,$Y$$\in$$T\overline{M}$.
Now we prove the following proposition.

Proposition 5.2. Let M be a ξ-vertical CR-submanifold of a Kenmotsu manifold $\overline{M}$.Then, the distribution D is integrable if the following are satisfied:

$$S(X,Y) \in D, \quad h(X,\phi Z) = h(\phi X, Z) \quad (63)$$

for any X,$Z$$\in$$D$.
Proof.The torsion tensor $S(X,Y)$ of the almost contact structure (ϕ,ξ,η,g) is given by

$$S(X,Y) = N_\phi(X,Y) + 2d\eta(X,Y)\xi = N_\phi(X,Y) + 2g(\phi X,Y)\xi. \quad (64)$$

Thus, we have

$$S(X,Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + 2g(\phi X, Y)\xi \tag{65}$$

for any X,$Y \in T\overline{M}$. Suppose that the distribution D is integrable.So,for X,$Y \in D$,$Q[X,Y] = 0$ and $\eta([X,Y]) = 0$ as $\xi \in D^{\perp}$.
If $S(X,Y) \in D$, then from equations (5.7) and (5.9) we have

$$[2g(\phi X, Y)\xi + \eta[X,Y]\xi + 4\phi\nabla_Y \phi X + 4\phi h(Y, \phi X) + 4\nabla_Y X + 4h(X,Y)] \in D \tag{66}$$

or

$$2g(\phi X, Y)Q\xi + \eta([X,Y])Q\xi + 4(\phi Q\nabla_Y \phi X + \phi h(Y, \phi X)$$
$$+ Q\nabla_Y X + h(X,Y)) = 0 \tag{67}$$

for any X,$Y \in D$ Replacing Y by ϕZ for $Z \in D$ in above equation, we get

$$2g(\phi X, \phi Z)Q\xi + 4(\phi Q\nabla_{\phi Z}\phi X + \phi h(\phi Z, \phi X)$$
$$+ Q\nabla_{\phi Z} X + h(X, \phi Z)) = 0 \tag{68}$$

Interchanging X and Z for X,$Z \in D$ in equation (5.13) and subtracting these relations,we obtain

$$\phi Q[\phi X, \phi Z] + Q[X, \phi Z] + h(Z, \phi X) - h(Z, \phi X) = 0 \tag{69}$$

for any X,$Y \in D$ and the assertion follows. Now, we prove the following proposition.
Proposition 5.3. Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$.Then

$$A_{\phi Y} Z - A_{\phi Z} Y = \frac{1}{3}\phi P[Y,Z] \tag{70}$$

for any Y,$Z \in D^{\perp}$.

Proof. For Y,$Z \in D^{\perp}$ and $X \in T(M)$, we get

$$\begin{aligned}
2g(A_{\phi Z} Y, X) &= 2g(h(X,Y), \phi Z) \\
&= g(h(X,Y), \phi Z) + g(h(X,Y), \phi Z) \\
&= g(\overline{\nabla}_X Y, \phi Z) + g(\overline{\nabla}_Y X, \phi Z) \\
&= g(\overline{\nabla}_X Y + \overline{\nabla}_Y X, \phi Z) \\
&= -g(\phi(\overline{\nabla}_X Y + \overline{\nabla}_Y X), Z) \\
&= -g(\overline{\nabla}_X \phi Y + \overline{\nabla}_Y \phi X + \eta(Y)\phi X + \eta(X)\phi Y, Z) \\
&= -g(\overline{\nabla}_X \phi Y, Z) - g(\overline{\nabla}_Y \phi X, Z) \\
&= g(\overline{\nabla}_Y Z, \phi X) + g(A_{\phi Y} Z, X)
\end{aligned}$$

The above equation is true for all $X \in T(M)$, therefore ,transvecting the vector field X both sides,we obtain

$$2A_{\phi Z}Y = A_{\phi Y}Z - \phi\overline{\nabla}_Y Z \tag{71}$$

for any Y,$Z \in D^{\perp}$. Interchanging the vector fields Y and Z,we get

$$2A_{\phi Y}Z = A_{\phi Z}Y - \phi\overline{\nabla}_Z Y \tag{72}$$

Subtracting equations (5.16) and (5.17), we get

$$A_{\phi Z}Y - A_{\phi Y}Z = \frac{1}{3}\phi P[Y, Z] \tag{73}$$

for any Y,Z $\in D^{\perp}$,completes the proof.
Theorem 5.4. Let M be a CR-submanifold of a Kenmotsu manifold $\overline{M}$. Then, the distribution $D^{\perp}$ is integrable if and only if

$$A_{\phi Z}Y - A_{\phi Y}Z = 0 \tag{74}$$

Proof. First Suppose that the distribution $D^{\perp}$ is integrable.Then $[Y, Z] \in D^{\perp}$ for any Y,$Z \in D^{\perp}$. Since P is a projection operator on D,so $P[Y, Z]$=0.Thus from equation (5.15) we get equation (5.20).Conversely, we suppose that equation (5.20) holds.Then using equation (5.15),we have $\phi P[Y, Z] = 0$ for any Y,$Z \in D^{\perp}$. Since rankϕ=2n.Therefore,either $P[Y, Z]$=0 or $P[Y, Z]=k\xi$.But $P[Y, Z]=k\xi$ is not possible as P is a projection operator on D.Thus,$P[Y, Z]$=0, which is equivalent to $[Y, Z] \in D^{\perp}$ for any Y,$Z \in D^{\perp}$ and hence $D^{\perp}$ is integrable.

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