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Conformal Vector fields on a locally projectively flat kropina metric

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In this paper, we study and characterize conformal vector fields on a Finsler manifold with Kropina metric of projectively isotropic flag curvature. Further, we prove that any conformal vector field on a non-Riemannian locally projectively flat Kropina metric of dimension $n \geq 3$ must be homothetic and completely determine conformal vector fields on a locally projectively flat Kropina metric.

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1. Introduction

The theory on conformal vector fields is one of the core contents of conformal geometry, which plays an important role in differential geometry and physics. (α, β)-metrics form a special and important class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta := b_i(x)y^i$ is a 1-form on M and $\phi(s)$ is a C^∞ positive function on some open interval. In particular, when $\phi(s) = \frac{1}{s}$, the Finsler metric $F = \frac{\alpha^2}{\beta}$ is called a Kropina metric. Kropina metrics, which were first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and investigated by V.K. Kropina [6], seem to be among the simplest nontrivial Finsler metrics with many interesting applications in physics etc [12].

Zermelo's navigation problem is to determine the shortest time paths for an object with constant internal force in R^2 under the influence of an external force. Later, Z. Shen discussed the navigation problem in a more general setting [1,2]. It is known that a Finsler metric is a Randers metric if and only if it is a solution of Zermelo's navigation problem on a manifold M with a Riemannian metric $h =$

$\sqrt{h_{ij}(x)y^iy^j}$ under the influence of an external force field $W = W^i(x)\frac{\partial}{\partial x^i}$ by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad W_0 := W_i y^i, \tag{1}$$

where $W_i := h_{ij}W^j$ and $\lambda := \|W\|_h < 1$. Here $\|W\|$ denotes the norm of W with respect to h. The condition $\|W\|_h < 1$ is essential for obtaining a positive definite Randers metric by Zermelo's navigation problem. In Ref[15] have shown that Kropina metrics are singular solutions of Zermelo's navigation problem. Let $h = \sqrt{h_{ij}(x)y^iy^j}$ be a Riemannian metric and $W^i(x)\frac{\partial}{\partial x^i}$ be a vector field with $\|W\|_h < 1$. Then the solution of the following Zermelo's navigation problem

$$h\left(x, \frac{y}{F(x,y)} - V_x\right) = 1, \quad y \in T_xM. \tag{2}$$

F is determined from (2) by h and W. Thus there is an one-to-one correspondence between a Kropina metric F and a pair (h, W) with $\|W\| = 1$ and it is easy to see that a Kropina metric can be regarded as the limit of the navigation problem for Randers metrics when $\|W\|_h \to 1$. We also call the pair (h, W) the navigation data of a Kropina metric F. Motivated by above researches and based on the characterizations of conformal vector fields on Kropina manifolds given in [1], we study Zermelo's navigation problem on a Kropina manifold $(M, F = \frac{\alpha^2}{\beta})$.

The flag curvature in Finsler geometry is a natural analogue of sectional curvature in Riemannian geometry and is an important Riemannian geometric quantity. For a Finsler manifold (M, F), the flag curvature $K = K(P, y)$ of F is a function of flag $P \in T_xM$ and flagpole $y \in T_xM$ at x with $y \in P$. A Finsler metric F is said to be of weakly isotropic flag curvature if $K = \frac{3\theta}{F} + \sigma$, where $\sigma = \sigma(x)$ is a scalar function and θ is a 1-form on M. In [9] classified locally projectively flat Kropina metrics with constant Ricci curvature and obtain Kropina metrics of zero flag curvature. G. Yang studied a class of singular (α, β)-metrics which are locally projectively flat with constant flag curvature in dimension $n = 2$ and $n \geq 3$ respectively and Kropina metrics which are projectively flat with constant flag curvature (see [15]). Z. Shen and Q. Xia studied conformal vector fields on a locally projectively flat Randers manifold (see [11]).

Recently, some progress has been made in the study of the conformal vector fields on a Finsler manifold [3,5,10]. In particular, we have completely determined all conformal vector fields on a Kropina metric of weakly isotropic flag curvature and constructed a On m-Kropina Finsler Metrics of Scalar Flag Curvature[3,14]. For a Kropina metric $F = \frac{\alpha^2}{\beta}$, if there is a closed 1-form η such that $\bar{F} := \frac{\alpha^2}{\bar{\beta}}(\bar{\beta} := \beta - \eta)$ is of weakly isotropic flag curvature, then $F = \frac{\alpha^2}{\beta}$ is projectively equivalent to $\bar{F} := \frac{\alpha^2}{\bar{\beta}}$ and hence it is of scalar flaag curvature. Kropina metrics $F = \frac{\alpha^2}{\beta}$ with such property are said to be of projectively isotropic flag curvature. Obviously, every locally projectively flat Kropina metrics or every Kropina metrics of weakly isotropic flag curvature is of projectively isotropic flag curvature and consequently of scalar flag curvature. The purpose of the present paper is to study and characterize conformal vector fields on a Kropina metrics of projectively isotropic flag curvature. In particular, we completely determine conformal fields on a locally projectively flat Kropina manifold.

2. Preliminaries

In this section, we characterize the navigation problem and some results on conformal vector fields on a Kopina manifold $(M; F = \frac{\alpha^2}{\beta})$.

Lemma 2.1

Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on a manifold M of dimension $n \geq 3$ with navigation data (h, W)[11]. Then F is of weakly isotropic flag curvature $K_F = \frac{3\theta}{F} + \sigma$ if and only if, at every point $x \in M$, there is a local coordinate system, in which h and $W = W^i \frac{\partial}{\partial x^i}$ are given by one of the following:

(1) $h = |y|$ is an Euclidean metric in R^n and $W = W^i \frac{\partial}{\partial x^i}$ is a nonzero constant vector field. In this case, $F = \frac{|y|^2}{2W_0}$ is a Minkowski metric with $K_F = 0$, where $W_0 = \langle W, y\rangle$ and $\langle , \rangle$ denotes Euclidean inner product in R^n.

(2)

$$h = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1 + \mu|x|^2}, \tag{3}$$

$$W = Qx + \mu\langle d, x\rangle x + d, \tag{4}$$

where μ is a positive constant, d is a nonzero constant vector with $|d| = 1$, and Q is a skew symmetric matrix with $Qd = 0$ and $Q^T Q + \mu d^T d = \mu E$, where T denotes the transpose of a matrix and E is an identity matrix. In this case, $||W||_h = 1$, $W_0 = \langle Qx + d, y\rangle 1 + \mu|x|^2$ and $F = \frac{h^2}{2W_0}$ is a Kropina metric with $K_F = \sigma = \mu > 0$ and $\theta = 0$.

We call the above expressions of h and W the local standard expression of F. With this, we can determine all conformal vector fields V with conformal factor $c(x)$ on a Kropina manifold (M, F) of weakly isotropic flag curvature when dim $M \geq 3$ [3]. In fact, in the same local coordinates for the local standard expression of F, V is given by one of the following [11]

(1) $V = xQ$, where Q and v are those in (2.8) with $vQ = 0 (v \neq 0)$. In this case, $c = 0$.

(2)

$$V = 2(\epsilon\sqrt{1 + \mu|x|^2} + \langle a, x\rangle)x - \frac{2|x|^2 a}{1 + \sqrt{1 + \mu|x|^2}}, \tag{5}$$

where ϵ is a constant and a is a nonzero constant vector in R^n. In this case, $c = \frac{\epsilon + \langle a, x\rangle}{\sqrt{1+\mu|x|^2}}$

(3)

$$V = 2\epsilon\sqrt{1+\mu|x|^2}x + xQ + d + \mu\langle x, d\rangle x, \tag{6}$$

where ϵ is a constant with $\delta\epsilon = 0$, Q and d are those in (2.8) with $\delta d = \mu\epsilon d = \epsilon Q = 0$. In this case, $c = \frac{\epsilon}{\sqrt{1+\mu|x|^2}}\epsilon$.

A vector field V on a Finsler manifold (M, F) is called a conformal vector field with a conformal factor $c = c(x)$ if the 1-parameter transformation φ_t generated by V is a conformal transformation on (M, F). In local coordinates, conformal vector fields V are characterized by

$$V_{i;j} + V_{j;i} + 2C^p_{ij}V_{p|q}y^q = 4cg_{ij} \tag{7}$$

where C_{ijp} are the coefficient of Cartan torsion C of F, $C^p_{ij} = g^{pq}C_{ijq}$, $V_i = g_{ij}V^j$ and "|" is the horizontal covariant derivative with respect to the Chern connection of F [3]. When $F = \frac{\alpha^2}{\beta}$ be a Kropina metric, conformal vector fields V are characterized by

$$V_{i;j} + V_{j;i} = 4ca_{ij}, \tag{8}$$

$$V^j b_{i;j} + b^j V_{j;i} = 2(2c - \rho)b_i. \tag{9}$$

where we use the Riemannian metric tensor a_{ij} to raise and lower the indices of V or b and ";" is the covariant derivative with respect to α. We can also express $F = \frac{\alpha^2}{\beta}$ in terms of the navigation data (h, W) by (2). It has been shown that V is conformal with respect to F if and only if

$$V_{i;j} + V_{j;i} = 4ch_{ij}, \tag{10}$$

$$V^j b_{i;j} + b^j V_{j;i} = 2(2c - \rho)W_i. \tag{11}$$

where we use the Riemannian metric tensor h_{ij} to raise and lower the indices of V or W and ";" is the covariant derivative with respect to h ([3], [4]).

3. Conformal Vector Field on Projectively flat Kropina Metric

In this section, we will study conformal vector fields on a Kropina manifold of projectively isotropic flag curvature and prove Theorem 3.1 and 3.2.

Theorem 3.1

Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold $M(n \geq 3)$ such that $\bar{F} = \frac{\alpha^2}{\beta}$ is a Kropina metric of weakly isotropic flag curvature and $\eta := \eta - \bar{\eta}$ is closed. Let $(\bar{h}, \bar{W})$ be the navigation data of $\bar{F}$ and V be a conformal vector field on (M, F) with conformal factor $c(x)$. Assume $\eta = \eta_i y^i \neq 0$ and V satisfies

$$V^j \eta_{i;j} + \eta^j V_{j;i} = 2(2c - \rho)\eta_i \tag{12}$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of Riemannian metric $\bar{h}$ and and $\rho = \rho(x)$ is a scalar function on M. Then V must be homothetic with respect to F. First, we need the following Lemmas.

Lemma 3.2

Let $\bar{F} = \frac{\alpha^2}{\bar{\beta}}$ and $F = \frac{\alpha^2}{\beta}$ be Kropina metrics on a manifold M. Let $\eta := \beta - \bar{\beta}$ and $(\bar{h}, \bar{W})$ be the navigation data for $\bar{F}$. Assume V is a vector field on M. Then each two of the following imply the third one.

(1) V is a conformal vector field on (M, F) with conformal factor $c(x)$;
(2) V is a conformal vector field on $(M, \bar{F})$ with conformal factor $c(x)$;
(3) $\eta = (\eta_i)$ satisfies

$$V^j \eta_{i;j} + \eta^j V_{j;i} = 2(2c - \rho)\eta_i \tag{13}$$

where we use $\bar{h}_{ij}$ to raise and lower the indices of V and η and ";" is the covariant derivative with respect to the Levi-Civita connection of $\bar{h}$.

Proof: We shall only prove that (1) and (3) imply (2). From [10, Corollary 2.1'], V is a conformal field with conformal factor $c(x)$ with respect to $F = \alpha + (\bar{\beta} + \eta)$ if and only if V satisfies

$$X_V(\alpha^2) = 4c\alpha^2, \tag{14}$$

$$X_V(\bar{\beta} + \eta) = 2(2c - \rho)(\bar{\beta} + \eta), \tag{15}$$

where $X_V = V^i \frac{\partial}{\partial x^i} + y^j \frac{\partial V^i}{\partial x^j} \frac{\partial}{\partial y^i}$ (see [10]) is a complete lift of V on $TM \backslash 0$. We have the following expressions for α and $\bar{\beta}$:

$$\alpha = \frac{\sqrt{\bar{\lambda}\bar{h}^2 + \bar{W}_0^2}}{\bar{\lambda}},$$

$$\bar{\beta} = -\frac{\bar{W}_0}{\bar{\lambda}},$$

where $\bar{\lambda} := 1 - \|\bar{W}\|^2_{\bar{h}}$. Equations (14) and (15) are rewritten as the following in terms of $(\bar{h}, \bar{W}, \eta)$:

$$\bar{\lambda}^2[X_V(\bar{h}^2) - 4c\bar{h}^2] + \bar{h}^2 X_V(\bar{\lambda}^2) = 2\bar{\lambda}\bar{W}_0[X_V(\bar{\eta}) - 2(2c - \rho)\eta]. \tag{16}$$

$$\bar{\lambda}^2[X_V(W_0) - 2(2c - \rho)W_0] + W_0 X_V(\bar{\lambda}^2) = -\bar{\lambda}^2[X_V(\eta) - 2(2c - \rho)\eta]. \tag{17}$$

If $X_V(\eta) - 2(2c - \rho)\eta = 0$, then (3.5) and (3.6) are reduced to

$$\bar{\lambda}^2[X_V(\bar{h}^2) - 4c\bar{h}^2] = -\bar{h}^2 X_V(\bar{\lambda}^2). \tag{18}$$

$$\bar{\lambda}^2[X_V(W_0) - 2(2c - \rho)W_0] = -W_0 X_V(\bar{\lambda}^2). \tag{19}$$

$$X_V(\eta) = 2(2c-\rho)\eta. \tag{20}$$

Observe that (18) and (19) are exactly (14) and (15) in [4], which are equivalent to $X_V(\alpha^2) = 4c\alpha^2$ and $X_V(\bar{\beta}) = 2(2c-\rho)\bar{\beta}$, that is, V is conformal with respect to $\bar{F} = \frac{\alpha^2}{\beta}$. Moreover,

$$X_V(\eta) = X_V(\eta_i y^i) = V^i \eta_{j;i} y^j + \eta^i V_{i;j} y^j. \tag{21}$$

Thus, $X_V(\eta) - 2(2c-\rho)\eta = 0$ is equivalent to (3.2). Consequently (1) and (3) imply (2). □

Lemma 3.3
Let $\bar{F} = \frac{\alpha^2}{\beta}$ be a Kropina metric on an n-dimensional manifold M with the navigation data $(\bar{h}, \bar{W})$. Suppose that $\eta = \eta_i y^i$ is a closed 1-form on M and $V = (V^i)$ is a vector field on M satisfying (13). Then either $c(x) = constant$ or $\eta = \nu(2c_0 - \rho)$, where $\nu = \nu(x)$ is a scalar function on M with $\nu_j(2c_i - \rho) = \nu_i(2c_j - \rho)$ and $(2c_0 - \rho) = (2c_i - \rho)y^i$ is a 1-form on M, here $(2c_i - \rho) := (2c_{x^i} - \rho)$ and $\nu_i := \nu_{x^i}$.

Proof: By assumption and (13), we have $\eta_{i;j} = \eta_{j;i}$ and $V = (V^i)$ satisfies

$$V^j \eta_{i;j} + \eta^j V_{j;i} = 2(2c-\rho)\eta_i. \tag{22}$$

Taking the covariant derivative on the both sides of (22), we get

$$V^j_{;k}\eta_{j;i} + V^j \eta_{j;i;k} + \eta^j_{;k} V_{j;i} + \eta^j V_{j;i;k} = 2(2c-\rho)\eta_{i;k} + 2(2c_k - \rho)\eta_i. \tag{23}$$

Exchanging the indices i, k in (23) yields

$$V^j_{;i}\eta_{j;k} + V^j \eta_{j;k;i} + \eta^j_{;i} V_{j;k} + \eta^j V_{j;k;i} = 2(2c-\rho)\eta_{k;i} + 2(2c_i - \rho)\eta_k. \tag{24}$$

observe that

$$V^j \eta_l \bar{R}_j{}^l{}_{ik} + V_l \eta^j \bar{R}_j{}^l{}_{ik} = V^j \eta^l \bar{R}_{jlik} + V^l \eta^j \bar{R}_{jlik} = 0, \tag{25}$$

where $\bar{R}_{jlik}$ is a Riemannian curvature tensor of $\bar{h}$. Subtracting (23) from (24) yields

$$(2c_k - \rho)\eta_i = (2c_i - \rho)\eta_k. \tag{26}$$

Here we have used the Ricci identity and (25).
Assume that $dc \neq 0$. It follows from (26) that there is a scalar function $\nu = \nu(x)$ such that

$$\eta_i = \nu(2c_i - \rho)$$

Further, $d\eta = 0$ implies that

$$\nu_j(2c_i - \rho) = \nu_i(2c_j - \rho).$$ □

Now we prove the main Theorem.3.1

Proof: We prove the theorem by contradiction. Assume that V is a conformal vector field on (M, F) with a non-constant conformal factor $c = c(x)$. By Lemma 3.2, V is also a conformal vector field $(M, \bar{F})$ with the same conformal factor $c = c(x)$. Since $\bar{F} = \frac{\alpha^2}{\beta}$ is of weakly isotropic flag curvature and $n \geq 3$, at any point, there is a local coordinate system $(U, (x^i))$, in which $\bar{h} = \sqrt{h_{ij}y^iy^j}$ and $\bar{W} = \bar{W}^i\frac{\partial}{\partial x^i}$ are given by (3) and (4). We have

$$\bar{h}_{ij} = \frac{\delta_{ij}}{1+\mu|x|^2} - \frac{\mu x^i x^j}{(1+\mu|x|^2)^2} \tag{27}$$

Its inveres $(\bar{h}^{ij})$ and the connection coefficients $\bar{\Gamma}^k_{ij}$ are respectively given by

$$\bar{h}^{ij} = (1+\mu|x|^2)(\delta_{ij} + \mu x^i x^j), \qquad \bar{\Gamma}^k_{ij} = -\frac{\mu(x_i\delta^k_j + x_j\delta^k_i)}{1+\mu|x|^2}. \tag{28}$$

Assume that $c(x) \neq$ *constant* on U. By (5) and (6), V is given by one of the following

(E1) $V = 2(\epsilon\sqrt{1+\mu|x|^2} + \langle a, x\rangle)x - \frac{2|x|^2 a}{1+\sqrt{1+\mu|x|^2}}$ $(a \neq 0)$. in this case, $c = \frac{\epsilon + \langle a,x\rangle}{\sqrt{1+\mu|x|^2}}$.

(E2) $V = 2(\epsilon\sqrt{1+\mu|x|^2}$ $(\mu \neq 0, \epsilon \neq 0)$. In this case, $c = \frac{\epsilon}{\sqrt{1+\mu|x|^2}}$. Moreover, by Lemma 3.3, there is a function v on M such that $\eta_i = \nu(2c_i - \rho) \neq 0$ with $\nu_j(2c_i - \rho) = \nu_i(2c_j - \rho)$. Consequently, there is a function $\sigma(x)$ on M such that $\nu_i = \sigma(2c_i - \rho)$. Noting that $c_{;i;j} = -\mu c\bar{h}_{ij}$ from Lemma 5.2.9 in [2] when $n \geq 3$. We have

$$\eta^i = \bar{h}^{ij}\nu c_j = v\sqrt{1+\mu|x|^2}(a^i - \mu\epsilon x^i),$$

$$\eta_{i;j} = \nu_j c_i - \mu c\nu\bar{h}_{ij} = \sigma c_i c_j - \mu c v\bar{h}_{ij}. \tag{29}$$

Case I: If V is given by (E1), then

$$V_i = \bar{h}_{ij}V^j = \frac{2cx_i}{1+\mu|x|^2} - \frac{2|x|^2 a_i}{(1+\mu|x|^2)(1+\sqrt{1+\mu|x|^2})}, \tag{30}$$

where the indices of a and x are raised and lowered by δ_{ij}. From (30) and (28), we have

$$V_{j;i} = 2c\bar{h}_{ij} - \frac{2(a_j x_i - a_i x_j)}{(1+\mu|x|^2)^2}, \tag{31}$$

Plugging (29) and (31) into (13) yields

$$Ax + (1+\tau)^{-1}Ba = 0, \tag{32}$$

$$A := -\mu(2c-\rho)\tau(\sigma D + 2(2c-\rho)\nu) - 2\nu|a|^2 + 2\mu\epsilon\nu\langle x, a\rangle, \tag{33}$$

$$B := \sigma\tau^2(1+\tau)D + 2\mu(2c-\rho)\nu\tau|x|^2 + 2\nu(1+\tau)(\langle x, a\rangle - \mu\epsilon|x|^2), \tag{34}$$

$$D := V^j(2c_j - \rho) = 2\epsilon\langle a, x\rangle + \frac{2\langle a, x\rangle^2}{\tau} - \frac{2|a|^2|x|^2}{\tau(1+\tau)} - \frac{2(2c-\rho)\mu(\epsilon + c)}{1+\tau}|x|^2, \tag{35}$$

where $\tau := \sqrt{1+\mu|x|^2}$. From (32) and $a \neq 0$, we get $A = 0$ and $B = 0$. If $\mu = 0$, then $A = 0$ implies $\nu = 0$, that is $\eta = 0$. This is impossible by assumption. Hence $\mu \neq 0$. Multiplying $\tau(1+\tau)$ on both sides of $A = 0$ and $2(2c-\rho)\mu$ on both sides of $B = 0$, and adding these two identities yield

$$\begin{aligned} &-\tau^2(1+\tau)\mu(2c-\rho)^2 - \tau(1+\tau)|a|^2 + \tau(1+\tau)\mu\epsilon\langle x, a\rangle + (2c - rho)^2\mu^2\tau|x|^2 \\ &+ (1+\tau)(2c-\rho)\mu(\langle x, a\rangle - \mu\epsilon|x|^2) = 0. \end{aligned} \tag{36}$$

By the irrationality of τ, (36) is decomposed as

$$\tau^2(-\mu(2c-\rho)^2 - |a|^2 + \mu\epsilon\langle x, a\rangle) + (2c-\rho)\mu(\langle x, a\rangle - \mu\epsilon|x|^2) = 0. \tag{37}$$

$$-\mu(2c-\rho)^2\tau^2 - |a|^2 + \mu\epsilon\langle x, a\rangle + (2c-\rho)^2\mu^2|x|^2 + (2c-\rho)\mu(\langle x, a\rangle - \mu\epsilon|x|^2) = 0. \tag{38}$$

(37) and (38) gives

$$\mu\langle a, x\rangle = |a|^2 + (2c-\rho)^2\mu. \tag{39}$$

Plugging (39) into (37) and using $\mu \neq 0$, $c \neq constant$ yield $\langle x, a\rangle - \mu\epsilon|x|^2 = 0$. By replacing x by $-x$ and adding these two identities, we get $\langle x, a\rangle = \mu\epsilon|x|^2 = 0$, which means $a = 0$. This is a contradiction with $a \neq 0$. Consequently, c is a constant.
Case II: If V is given by (E2), then

$$V_i = \frac{2(2c-\rho)x_i}{1+\mu|x|^2}, \quad V_{i;j} = 2c\bar{h}_{ij}. \tag{40}$$

In this way as Case I, we get $\bar{A}x = 0$, which means $\bar{A} = 0$, where

$$\bar{A} := \sigma(2c-\rho)\mu^2(\epsilon + c)|x|^2 - (1+\tau)(2c-\rho)^2\mu\nu. \tag{41}$$

Since $\mu \neq 0$ and c is not constant, $\bar{A} = 0$ means $\nu = 0$ by the irrationality of τ, which is impossible because of $\eta \neq 0$. This complete the proof. □

Theorem 3.2
Let F and $\bar{F}$ be as in Theorem 3.1. Assume $(\bar{h}, \bar{W})$ is the navigation data of $\bar{F}$ given by (3)-(4). Let V be a vector field on R^n given by one of the following

(1) $V = xQ$, where Q and v are those in (2.3) with $vQ = 0 (v \neq 0)$;
(2) $V = 2\epsilon\sqrt{1+\mu|x|^2}x + Qx + \mu\langle d, x\rangle x + d$ where ϵ is a constant with $\epsilon\mu = \epsilon\delta = 0$, and Q and d are those in (2.3) with $\delta d = \epsilon Q = 0$.

If there is a function $f = f(x)$ on M such that $\eta = df \neq 0$, which satisfies

$$V^j f_{x^j} - 2(2c-\rho)f = k, \tag{42}$$

where k is a constant, then V is a homothetic vector field of F with dilation ϵ. Conversely, if V is a homothetic vector field of F with dilation ϵ and $\eta = df \neq 0$ satisfying (42), then V must be given by (1) or (2) above.

We need the following Lemma to prove the Theorem 3.3

Lemma 3.4

Assume that the vector field V in Lemma 3.2 is homothetic, that is $c =$ constant and $\eta = df$ for some scalar function $f = f(x)$. Then (13) is equivalent to the following equation

$$V^j f_{x^j} - 2(2c - \rho)f = k, \tag{43}$$

where $k = constant$.

Proof: By assumption, $\eta_i = f_{x^i}$. Thus $\eta_{i;j} = \eta_{j;i}$. Assume that V and η satisfy (13). Then

$$\begin{aligned}(V^j\eta_j)_{;i} &= V^j\eta_{j;i} + V^j_{;i}\eta_j \\ &= V^j\eta_{i;j} + \eta^j V_{j;i} \\ &= 2(2c-\rho)\eta_i = 2(2c-\rho)f_{x^i}.\end{aligned}$$

Thus

$$(V^j f_{x^j} - 2(2c-\rho)f)_{x^i} = 0.$$

We conclude that $V^j f_{x^j} - 2(2c-\rho)f = constant$. The converse is trivial. □

Now we prove the main Theorem 4.2,

Proof: By the assumption and

$$V = 2(\epsilon\sqrt{1+\mu|x|^2} + \langle a, x\rangle)x - \frac{2|x|^2 a}{1+\sqrt{1+\mu|x|^2}}, \tag{44}$$

$$V = 2\epsilon\sqrt{1+\mu|x|^2}x + xQ + d + \mu\langle x, d\rangle x, \tag{45}$$

V is homothetic with respect to $\bar{F}$ with dilation ϵ. Thus V is homothetic with respect to F with dilation ϵ from Lemma 4.4 and Lemma 3.2. Conversely, if V is homothetic with respect to F with dilation ϵ and η satisfies (42), then by Lemma 4.2 and Lemma 3.2, V is also homothetic with respect to $\bar{F}$ with dilation ϵ and (1) and (2) in Theorem 3.3 follows from (44) and (45) directly. □

Corollary 3.1.

Let $F = \frac{\alpha^2}{\beta}(\beta \neq 0)$ be a locally projectively flat Kropina metric on an n-dimensional manifold M. Suppose V is a conformal vector field on (M, F) and dim $M \geq 3$. Then V must be homothetic. In this case,

$$V = 2\epsilon\sqrt{1+\mu|x|^2}x + xQ + d + \mu\langle x, d\rangle x,$$

where δ, μ are constants with $\delta\mu = 0$, Q is a constant skew symmetric matrix and d is a constant vector in R^n.

Consider a special Kropina metric $F = \frac{\alpha^2}{\beta}$ on R^n

$$\alpha = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1+\mu|x|^2}, \tag{46}$$

$$\beta = df \neq 0, \tag{47}$$

where $f = f(x)$ is a scalar function. F is projectively flat. Let

$$\Phi = f_{i|j}y^i y^j, \quad \Psi = f_{i|j|k}y^i y^j y^k. \tag{48}$$

where $f_{i|j}$ and $f_{i|j|k}$ are the coe±cients of the covariant derivatives of f with respect to α. Then the flag curvature of F is given by

$$K_F(x, y) = \frac{\mu\alpha^2}{F^2} - \frac{\Psi}{2F^3} + \frac{3\Phi^2}{4F^2}. \tag{49}$$

(cf. see (6.10) in [2]). Let V be a conformal vector field on (M, F) with conformal factor c. Then, V is homothetic and is given by (1). By [8], $\tilde{F}$ generated from (F, V) by solving (2) is of scalar flag curvature $K_{\tilde{F}}(x; y) = K_F(x, y - \tilde{F}V) - c^2$. Thus one obtains a series of Kropina metric of scalar flag curvature. In general, such $\tilde{F}$ is not locally projectively flat, not of constant flag curvature. To see this, we consider a more special case of following lemma.

Lemma 3.5
Let $F = \frac{\alpha^2}{\beta}$ be Kropina metric with $||\beta||_\alpha = 1$ and define $\tilde{F} = F + \eta$, where η is a closed 1-form with $||\eta||_\alpha$ sufficiently small[14].

- If F is of scalar flag curvature, then $\tilde{F}$ is also a Kropina metric of scalar flag curvature.
- Let F be of constant flag curvature. Then $\tilde{F}$ is locally projectively flat if and only if F is flat-parallel, or equivalently, $\tilde{F}$ can be locally written in the form

$$\tilde{F} = \frac{|y|}{y^1} + \eta. \tag{50}$$

In the above lemma 3.5 (ii) easily follows from a known result , since therein proved that a locally projectively flat Kropina metric with constant flag curvature is flat-parallel. By lemma 3.5 (ii), we can easily obtain a family of Kropina metrics which are of scalar flag curvature but are neither locally projectively flat nor of constant flag curvature in general. Take $\eta = \langle x, y\rangle$ with x close to origin, and then $\tilde{F}$ in (50) is a projectively flat Kropina metric with the flag curvature given by

$$K = \frac{3}{4}\left\{\frac{|y|^4(y^1)^4}{(\eta y^1 + |y|^2)^4}\right\}. \tag{51}$$

Additionally, using Corollary 4.3 [14] and a warped product method, we obtain a family of Kropina metrics which are locally projectively flat (see Proposition 5.2 [14]).

4. Conclusion

We have studied the Conformal vector fields on locally projectively flat Kropina metrics. In the present paper, Theorem 3.1 shows that V must be homothetic with respect to F and $\bar{F}$ respectively in this case. Thus, one can determine the homothetic vector fields on (M, F) from those on $(M, \bar{F})$. Further, in Theorem 3.1, since $\bar{F}$ is of weakly isotropic flag curvature and dim $M \geq 3$, at any point, there is a local coordinate system $(U, (x^i))$, in which $\bar{h}$ is of constant sectional curvature μ and $\bar{W}$ is a conformal vector field with conformal factor $\sigma(x)$ with respect to $\bar{h}$ according to [11]. In particular, if $\bar{\beta} = 0$ in Theorem 1.1, then $\eta = \beta$ is closed and $\bar{F} = \alpha$ is of isotropic sectional curvature (=constant if $dimM \geq 3$). Thus $F = \frac{\alpha^2}{\eta}$ is projectively flat. Since $\bar{F} = \bar{h} = \alpha$ is a Riemannian metric and V is conformal with respect to F, (12) holds by (9) and V is conformal with respect to α by Lemma 3.1.
Under the condition (12), V is a conformal vector field on (M, F) with the conformal factor $c(x)$ if and only if V is a conformal vector field on $(M, \bar{F})$ with the same conformal factor $c(x)$, which is regarded as the geometric meaning of the equation (12) (see Lemma 3.1).

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